Optimal Control of SARS Disease via Fractional Model

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ABSTRACT

Severe acute respiratory syndrome (SARS) is a very dangerous disease that affects the human respiratory system. In this article, we discuss the optimal control of this disease via a fractional SVEIR epidemic model together with two control variables (treatment and vaccination). For this purpose, we first design a fractional optimal control problem and then apply Pontryagin's minimal principle in a fractional version to find the optimal control. Also, the forward and backward fractional Euler methods (FEM) are used to solve the state and co-state equations, respectively. The results gave a new treatment and vaccine strategy for breaking dawn and preventing the spread of SARS.

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1. Introduction

Pathogens, which can include bacteria, fungi, viruses, and other microorganisms, are the biological agents that are responsible for the transmission of infectious diseases. Pathogens can also cause other diseases [1-5]. There are many different channels through which infectious diseases can be passed on from one person to another. Bodily fluids, faeces, and SARS-infected equipment are examples of these channels. [6]. Contact with an infected person (I) can make a susceptible person (S) get sick, which can then transmit the disease to other individuals who are susceptible. This occurrence is evidence that a disease has spread throughout an entire society or country, which can lead to an increase in the mortality rate in a relatively short amount of time. The World Health Organization (WHO) says that infectious diseases kill about $1/6$ of all people around the world and are the second leading cause of death [7]. *SARS* is a virus-borne infectious disease. Fever, shortness of breath, dry cough and other pneumonia-like symptoms are among the clinical symptoms of *SARS* [8]. *SARS* is a disease that spreads quickly, progresses, and is fatal. The majority of people who contract *SARS* are between the ages of 25 and 70, but it can also affect children under the age of 15 [9]. The first case of *SARS* was discovered in Guangdong province, China, in November 2002. The *SARS* virus also spread quickly across the continent, particularly in Asia and the Pacific. In March of 2003, the WHO reported a disease known as *SARS* that was caused by a virus known as *SARS-CoV* or coronavirus [8]. This led to 774 deaths and 8098 people getting sick with the *SARS-CoV* virus [10].

Infectious diseases have a big effect on the health of a country or population, so it's important to understand how they spread, come up with plans to stop them, and take into account the costs of doing so. In this case, we need a mathematical model to show how diseases will spread in the future based on information from the present. To comprehend the mechanism of spreading the disease and how procedures can be connected with disease control programs, mathematical modeling is necessary. The field of mathematics that was developed to determine the most effective means of controlling dynamic systems is known as optimal control theory. When optimal control theory is used, the goal is to figure out how effective different policies and control measures are and how much they cost [11]. Vaccination is extremely popular as a preventative measure. WHO-approved vaccinations are expected to save the lives of approximately 2-3 million people each year and prevent 25 diseases. There has been a rise in the use of vaccination and other control policies, such as quarantine, isolation, screening, and treatment, in recent decades [7].

In recent few years, Kumar and Srivastava included control variables in the model of the SVIR epidemic in an effort to halt the progression of the disease. In an attempt to stop the spread of disease, people also need to get treated and vaccinated. The optimal control theory was always used by scholars to implement an efficient economic strategy to reduce costs and ensure optimal management of treatment and vaccination to control and stop the spread of disease [12]. Kumar and Srivastava's research also looks at the costs because health or government institutions may not have enough money to pay for it [7]. Mathematical modeling was used in the past to try past to try to figure out how an anti-*SARS* vaccine might affect people, but it wasn't perfect. Further results show that an imperfect anti-*SARS* vaccine is still able to stop the spread of *SARS* in the public, but only if the vaccine is at least 75% effective [13].

Fractional calculus plays the main role in designing real-life models, especially biological systems. In fact, fractional derivatives possess three main properties that distinguish them from the classical derivative. The first is the mathematical model described by the fractional derivative, which takes the historical behaviour of the system into account. The second is that the region of stability of the fractional model is greater than the region of stability of the classical model. The third is the classical derivative, which is a local operator, but the fractional derivative is a nonlocal operator. This gives fractional derivatives more flexibility to describe more complicated real-life systems [14, 15, 16, 17, 18, 19, 20]. The goal of having a control function in this work is to decrease the number of exposed and infected subpopulations so that the *SARS* disease doesn't spread too far and to reduce the cost of putting control in place. Problems with optimal control were solved with Pontryagin's minimum principle combined with the FEM. The results of the simulations are also looked at to figure out the best ways to stop the spread of the SARS disease.

The following outline is the structure of this paper: Section two provides a general formulation of the fractional *SARS* Epidemic Model. The application of the maximal Ponntryagin principle has been discussed in Section 3. The numerical simulation results and their discussion are presented in section four. Section five summarizes the conclusions.

2. Formulation of the fractional SARS Epidemic Model

The *SVIR* model serves as the basis for the development of the *SVEIR* model, which incorporates subpopulation exposed *E*. According to the findings that Li et al. [12] presented, the subpopulation model *E* hypothesises that susceptible individuals who become infected with the disease on account of coming into exchange with an individual who already has the disease will go into the exposed *E* subpopulation at some point during the exposure period. When it's over, the exposure

period, the individual starts to display clinical symptoms and gains the disease's capacity to spread, becoming a member of the infected subpopulation *I*. This happens after the end of the exposure period. In addition to this, the *SVEIR* model makes the assumption that individuals who have been infected with *SARS* are unable to recover on their own through natural means. This presumption originates from the model that was developed as a result of Huang's research1. The incorporation of these hypotheses led to the conclusion that there is no natural recovery rate among the infected individuals. This conclusion was reached because of the addition of these assumptions.

This investigation into the spread of the *SARS* disease focused on a single population that had been subdivided into five different subpopulations: a susceptible *S*, a vaccinated *V*, an exposed *E*, an infected *I*, and a recovered *R*. In addition, a variety of research approaches are utilized in order to accomplish the goals. There are many interesting definitions of fractional derivatives in fractional calculus[13], but for this purpose, we will use the famous Caputo derivatives due to their advantage on initial value problems.

Definition 1 [21] The fractional integral of order $0 < \alpha < 1, t > 0$ is defined by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx \tag{1}$$

Definition 2 [21] Let $n-1 < \alpha < n$, the Caputo fractional derivative of order α is given by

$${}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(x)}{(t-x)^{\alpha+1-n}} dx \tag{2}$$

The model for the broadcast of the SARS disease is formulated as follows of fractional differential equations:

$$\begin{aligned}
&{}_0^C D_t^{\alpha} S(t) = \mu^{\alpha} - \beta_1^{\alpha} SI - \gamma_1^{\alpha} S - \mu^{\alpha} S,\\
&{}_0^C D_t^{\alpha} V(t) = \gamma_1^{\alpha} S - \beta_2^{\alpha} VI - \gamma_2^{\alpha} V - \mu^{\alpha} V,\\
&{}_0^C D_t^{\alpha} E(t) = \beta_1^{\alpha} SI + \beta_2^{\alpha} VI - \varphi^{\alpha} E - \mu^{\alpha} E,\\
&{}_0^C D_t^{\alpha} I(t) = \varphi^{\alpha} E - \gamma_t^{\alpha} I - \mu^{\alpha} I,\\
&{}_0^C D_t^{\alpha} R(t) = \gamma_2^{\alpha} V + \gamma_t^{\alpha} I - \mu^{\alpha} R.
\end{aligned} \tag{3}$$

All of the parameters in system (3) of the SVEIR model are positive constants. The fractional SVEIR model's parameters are described in Table 1.

Table 1: Fractional SVEIR parameters and meaning [7]

Parameter	Description

μ	The natural rate of birth and death.
β_1	Disease transmission rate from $I(t)$ to $S(t)$.
γ_1	The constant vaccination rate in $S(t)$.
β_2	The rate of disease transmission from *SARS*-infected to unvaccinated individuals.
γ_2	The vaccinated person's immunity so they recover during or after vaccination.
φ	Exposure-infection rate.
γ_t	Treatment rate of $I(t)$.

Treatment and vaccination have been chosen as control policies because vaccination is simple to obtain and put into practice, and these policies have the backing of the WHO [7]. The *SARS* model, which was previously explicated in a model system (3), was altered in this study in order to express it in a model system (4) with treatment and vaccination as a control strategy, which will be covered in the next section.

3. The optimal control problem of fractional model of SARS

The theory of optimal control is widely used to determine the extreme value of an objective function satisfying certain differential equations. In this section, optimal control theory is used to determine the best drug treatments and vaccination as a function of time. The *SARS* disease spread model with treatment and vaccination control strategies is derived from equations (3) and the following statements:

1. Anti-*SARS* vaccines are given to susceptible subpopulations to prevent disease transmission. From an economic point of view, getting vaccinated over and over again might not be a good idea for some diseases. In addition, administering vaccines to a large population is expensive and difficult, as it is difficult to reach all susceptible individuals. Vaccination policies should be limited in scope to meet predetermined objectives because health agencies have limited resources and time to do so. That's why we assume that vaccination control is limited, which is to say, $0 \le u_1(t) \le 1$ [7].

2. Infected subpopulations are treated to reduce disease burden and spread. Diagnosis, hospitalization, drug administration, and other medical services are all part of treatment programs. Treatment programs for infected individuals, like vaccination programs, are expensive and must be kept to a minimum. Therefore, γ_t^{α} on model (3) become as treatment control $u_2(t)$ and assumed that $0 \le u_2(t) \le 1$ [7].

Consequently, treatment and vaccination control of fractional model of *SARS* disease spread is given by:

$$\begin{aligned}
&{}_0^C D_t^\alpha S(t) = \mu^\alpha - \beta_1^\alpha SI - u_1(t)S - \mu^\alpha S,\\
&{}_0^C D_t^\alpha V(t) = u_1(t)S - \beta_2^\alpha VI - \gamma_2^\alpha V - \mu^\alpha V,\\
&{}_0^C D_t^\alpha E(t) = \beta_1^\alpha SI + \beta_2^\alpha VI - \varphi^\alpha E - \mu^\alpha E,\\
&{}_0^C D_t^\alpha I(t) = \varphi^\alpha E - u_2(t)I - \mu^\alpha I,\\
&{}_0^C D_t^\alpha R(t) = \gamma_2^\alpha V + u_2(t)I - \mu^\alpha R.
\end{aligned} \tag{4}$$

In other words, we're looking for the best control u so that.

$$\Im(u_1^*(t),u_2^*(t)) = \min\left\{\Im(u_1(t),u_2(t)) : 0 \le u_1,u_2 \le 1, 0 \le t \le T_f\right\} \tag{5}$$

Consider the cost function as follow:

$$\Im(u) = \int_0^{T_f} [E(t) + I(t) + A_1u_1^2(t) + A_2u_2^2(t)]dt \tag{6}$$

Where $A_1 \ge 0$ and $A_2 \ge 0$ represent the weight of vaccination and treatment, respectively, that should be minimized. Also, the Hamiltonian functional H is given as follows:

$$\begin{aligned}
H(S,V,E,I,R,u_1,u_2,\lambda,t) &= E + I + A_1u_1^2(t) + A_2u_2^2(t)\\
&+ \sum_{j=1}^{5}\lambda_j({}_0^C D_t^\alpha S(t), {}_0^C D_t^\alpha V(t), {}_0^C D_t^\alpha E(t), {}_0^C D_t^\alpha I(t), {}_0^C D_t^\alpha R(t))
\end{aligned} \tag{7}$$

Now, we can derive the necessary conditions from Eq. (7) as follows as in [22]

$${}_t^C D_{T_f}^\alpha \lambda_1(t) = \frac{\partial H(t)}{\partial S(t)} \tag{8}$$

$${}_t^C D_{T_f}^\alpha \lambda_2(t) = \frac{\partial H(t)}{\partial V(t)} \tag{9}$$

$${}_t^C D_{T_f}^\alpha \lambda_3(t) = \frac{\partial H(t)}{\partial E(t)} \tag{10}$$

$${}_t^C D_{T_f}^\alpha \lambda_4(t) = \frac{\partial H(t)}{\partial I(t)} \tag{11}$$

$${}_t^C D_{T_f}^\alpha \lambda_5(t) = \frac{\partial H(t)}{\partial R(t)} \tag{12}$$

And $\lambda_j(T_f) = 0, \forall j = 1,2,...,5$ are the Lagrange multipliers.

Theorem 1: Consider u_1^* and u_2^* are optimal controls with the corresponding state S^*, V^*, E^*, I^* and R^* then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 satisfies the following:

$$ {}_t^C D_{T_f}^{\alpha} \lambda_1(t) = (\lambda_3 - \lambda_1)\beta_1^{\alpha} I^* + (\lambda_2 - \lambda_1)u_1(t) - \lambda_1 \mu^{\alpha} $$

$$ {}_t^C D_{T_f}^{\alpha} \lambda_2(t) = (\lambda_3 - \lambda_2)\beta_2^{\alpha} I^* - \lambda_2(\gamma_2^{\alpha} + \mu^{\alpha}) + \lambda_5 \gamma_2^{\alpha} $$

$$ {}_t^C D_{T_f}^{\alpha} \lambda_3(t) = 1 + (\lambda_4 - \lambda_3)\varphi^{\alpha} - \lambda_3 \mu^{\alpha} $$

$$ {}_t^C D_{T_f}^{\alpha} \lambda_4(t) = 1 + (\lambda_3 - \lambda_1)\beta_1^{\alpha} S^* + (\lambda_3 - \lambda_2)\beta_2^{\alpha} V^* + (\lambda_5 - \lambda_4)u_2(t) - \lambda_4 \mu^{\alpha} $$

$$ {}_t^C D_{T_f}^{\alpha} \lambda_5(t) = -\lambda_5 \mu^{\alpha} $$

With transversality conditions $\lambda_j(T_f) = 0, \forall j = 1,2,...,5$ Furthermore, the optimal controls u_1^* and u_2^* are given by:

$$ u_1^*(t) = \min\{\max\{0, \frac{(\lambda_2 - \lambda_1)}{2A_1} S^*(t)\}, 1\} $$

$$ u_2^*(t) = \min\{\max\{0, \frac{(\lambda_5 - \lambda_4)}{2A_2} I^*(t)\}, 1\} $$

Proof: By definition the Hamiltonian function H , we get

$$ \begin{aligned} H(S,V,E,I,R,u_1,u_2,\lambda,t) = E + I + A_1 u_1^2(t) + A_2 u_2^2(t) + \lambda_1 {}_0^C D_t^{\alpha} S(t) + \lambda_2 {}_0^C D_t^{\alpha} V(t) \\ + \lambda_3 {}_0^C D_t^{\alpha} E(t) + \lambda_4 {}_0^C D_t^{\alpha} I(t) + \lambda_5 {}_0^C D_t^{\alpha} R(t) \end{aligned} \quad (13) $$

Then

$$ \begin{aligned} H(S^*,V^*,E^*,I^*,R^*,u_1^*,u_2^*,t) = E^* + I^* + A_1 u_1^{*2}(t) + A_2 u_2^{*2}(t) + \lambda_1 {}_0^C D_t^{\alpha} S^*(t) + \lambda_2 {}_0^C D_t^{\alpha} V^*(t) \\ + \lambda_3 {}_0^C D_t^{\alpha} E^*(t) + \lambda_4 {}_0^C D_t^{\alpha} I^*(t) + \lambda_5 {}_0^C D_t^{\alpha} R^*(t) \end{aligned} \quad (14) $$

$$ \begin{aligned} H(S^*,V^*,E^*,I^*,R^*,u_1^*,u_2^*,t) = E^* + I^* + A_1 u_1^{*2}(t) + A_2 u_2^{*2}(t) + \lambda_1(\mu^{\alpha} - \beta_1^{\alpha} S^* I^* - u_1(t)S^* - \mu^{\alpha} S^*) \\ + \lambda_2(u_1(t)S^* - \beta_2^{\alpha} V^* I^* - \gamma_2^{\alpha} V^* - \mu^{\alpha} V^*) + \lambda_3(\beta_1^{\alpha} S^* I^* + \beta_2^{\alpha} V^* I^* - \varphi^{\alpha} E^* \\ - \mu^{\alpha} E^*) + \lambda_4(\varphi^{\alpha} E^* - u_2(t) I^* - \mu^{\alpha} I^*) + \lambda_5(\beta_2^{\alpha} V^* + u_2(t) I^* - \mu^{\alpha} R) \end{aligned} $$

(15)

By using Pontryagin's minimum principle with we can get

$$ \begin{aligned} {}_t^C D_{T_f}^{\alpha} \lambda_1(t) &= (\lambda_3 - \lambda_1)\beta_1^{\alpha} I^* + (\lambda_2 - \lambda_1)u_1(t) - \lambda_1 \mu^{\alpha} \\ &= \frac{\partial H(t)}{\partial S^*(t)} \end{aligned} \quad (16) $$

$$
\begin{aligned}
{}_{t}^{C}D_{T_f}^{\alpha}\lambda_2(t) &= (\lambda_3-\lambda_2)\beta_2^{\alpha}I^{*} - \lambda_2(\gamma_2^{\alpha}+\mu^{\alpha}) + \lambda_5\gamma_2^{\alpha} \\
&= \frac{\partial H(t)}{\partial V^{*}(t)}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
{}_{t}^{C}D_{T_f}^{\alpha}\lambda_3(t) &= 1+(\lambda_4-\lambda_3)\varphi^{\alpha} - \lambda_3\mu^{\alpha} \\
&= \frac{\partial H(t)}{\partial E^{*}(t)}
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
{}_{t}^{C}D_{T_f}^{\alpha}\lambda_4(t) &= 1+(\lambda_3-\lambda_1)\beta_1^{\alpha}S^{*} + (\lambda_3-\lambda_2)\beta_2^{\alpha}V^{*} + (\lambda_5-\lambda_4)u_2(t) - \lambda_4\mu^{\alpha} \\
&= \frac{\partial H(t)}{\partial I^{*}(t)}
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
{}_{t}^{C}D_{T_f}^{\alpha}\lambda_5(t) &= -\lambda_5\mu^{\alpha} \\
&= \frac{\partial H(t)}{\partial R^{*}(t)}
\end{aligned} \tag{20}
$$

And the transversality conditions $\lambda_1(T_f)=\lambda_2(T_f)=\lambda_3(T_f)=\lambda_4(T_f)=\lambda_5(T_f)=0$. The optimal control u_1^{*} can be obtained from the condition

$$\frac{\partial H(t)}{\partial u_1^{*}(t)} = 0 \tag{21}$$

$$2A_1u_1^{*}(t) + (\lambda_2-\lambda_1)S^{*} = 0 \tag{22}$$

$$u_1^{*}(t) = \frac{(\lambda_2-\lambda_1)}{2A_1}S^{*}(t) \tag{23}$$

Since $0 \le u_1 \le 1$ then we can rewrite u_1^{*} in the Eq. (23) as follow

$$u_1^{*}(t) = \min\{\max\{0, \frac{(\lambda_2-\lambda_1)}{2A_1}S^{*}(t)\}, 1\} \tag{24}$$

Also, the optimal control u_2^{*} can be obtained from the condition

$$\frac{\partial H(t)}{\partial u_2^{*}(t)} = 0 \tag{25}$$

$$2A_2u_2^{*}(t) + (\lambda_5-\lambda_4)I^{*} = 0 \tag{26}$$

$$u_2^{*}(t) = \frac{(\lambda_5-\lambda_4)}{2A_2}I^{*}(t) \tag{27}$$

Since $0 \le u_2 \le 1$ then we can rewrite u_1^{*} in the Eq. (27) as follow

$$u_2^{*}(t) = \min\{\max\{0, \frac{(\lambda_5-\lambda_4)}{2A_2}I^{*}(t)\}, 1\} \tag{28}$$

4. Numerical Simulations

In this section, the results of numerical simulations are implemented to study the impact of control policy strategies on the dynamics of the disease as well as the costs associated with putting those strategies into action. Numerical simulations of systems (4) and Eq. (16)- Eq. (20) use the forward and backward Euler method in the MAPLE software. Now, we list the numerical simulation parameters in Table 2.

Table 2: Fractional SVEIR parameters [7]

Parameter	Value
μ	0.00005
β_1	0.3
β_2	0.01
γ_2	0.04
φ	0.02
u_1	$0-1$ variable
u_2	$0-1$ variable

In addition to this, we will use the suggested values for the initial conditions of the state variables that are presented in Table 3.

Table 3: Fractional *SVEIR* initial values [7]

Variable	Initial values
$S(t)$	0.8
$V(t)$	0.04
$E(t)$	0.08
$I(t)$	0.04
$R(t)$	0.04

To assess the utility of control policy, two control strategies are implemented, namely strategy 1: implementation of a no-vaccination and no-treatment policy to study the behaviour of SARS spread. While strategy 2 is the implementation of a treatment and vaccination policy to control the disease's spread. To find the optimal treatment and vaccination strategy, we construct the

following algorithm based on applying the forward and backward Euler method to solve state Eq. (4) and co-state equations Eq. (16)-Eq. (20), respectively, and on the optimal controls law in Eq. (24) and Eq. (28).

Algorithm 1:

Step 1: Insert the values of fractional order α, and the biological parameters $\mu,\beta_1,\beta_2,\gamma_2$ and φ. Also, insert the initial conditions of $S(0),V(0),E(0),I(0),R(0)$ and terminal conditions $\lambda_1(N)=\lambda_2(N)=\lambda_3(N)=\lambda_4(N)=\lambda_5(N)=0$.

Step 2: Suppose the time interval is $[0,T_f]$ and compute the step size $h=\frac{T_f}{N}$, where N is positive integer number.

Step 3: Set $u_1(\kappa h)=u_2(\kappa h)=0$, for all $\kappa=0,1,...,N$.

Step 4: Compute the coefficients $C_{j,\kappa}$ as follows: $C_{j,\kappa}=\frac{h^\alpha}{\Gamma(\alpha+1)}((j+1-\kappa)^\alpha-(j-\kappa)^\alpha)$, for all $j=\kappa,...,N-1$, and $\kappa=0,...,N-1$.

Step 5: Compute the coefficients $B_{j,\kappa}$ as follows: $B_{j,\kappa}=\frac{h^\alpha}{\Gamma(\alpha+1)}((\kappa-j)^\alpha-(\kappa-j-1)^\alpha)$, for all $j=0,1,...,\kappa-1$, and $\kappa=1,2,...,N$.

Step 6: For all $\kappa=1,2,...,N$, compute $S(\kappa h),V(\kappa h),E(\kappa h),I(\kappa h)$, and $R(\kappa h)$ by applying forward fractional Adams method as follows:

$$S(\kappa h)=S(0)+\sum_{j=0}^{\kappa-1}B_{j,\kappa}[\mu^\alpha-\beta_1^\alpha S(jh)I(jh)-u_1(jh)S(jh)-\mu^\alpha S(jh)]$$

$$V(\kappa h)=V(0)+\sum_{j=0}^{\kappa-1}B_{j,\kappa}[u_1(jh)S(jh)-\beta_2^\alpha V(jh)I(jh)-\gamma_2^\alpha V(jh)-\mu^\alpha V(jh)]$$

$$E(\kappa h)=E(0)+\sum_{j=0}^{\kappa-1}B_{j,\kappa}[\beta_1^\alpha S(jh)I(jh)+\beta_2^\alpha V(jh)I(jh)-\varphi^\alpha E(jh)-\mu^\alpha E(jh)]$$

$$I(\kappa h)=I(0)+\sum_{j=0}^{\kappa-1}B_{j,\kappa}[\varphi^\alpha E(jh)-u_2(jh)I(jh)-\mu^\alpha I(jh)]$$

$$R(\kappa h)=R(0)+\sum_{j=0}^{\kappa-1}B_{j,\kappa}[\beta_2^\alpha V(jh)+u_2(jh)I(jh)-\mu^\alpha R(jh)]$$

Step 7: For all $\kappa=N-1,N-2,...,0$, compute $\lambda_1(\kappa h),\lambda_2(\kappa h),\lambda_3(\kappa h),\lambda_4(\kappa h)$, and $\lambda_5(\kappa h)$ by applying backward fractional Adams method as follows:

$$\lambda_1(\kappa h) = \lambda_1(T_f) - \sum_{j=\kappa}^{N-1} C_{j,\kappa}[(\lambda_3((j+1)h) - \lambda_1((j+1)h))\beta_1^{\alpha} I^*((j+1)h) + (\lambda_2((j+1)h) - \lambda_1((j+1)h))u_1((j+1)h) - \lambda_1((j+1)h)\mu^{\alpha}]$$

$$\lambda_2(\kappa h) = \lambda_1(T_f) - \sum_{j=\kappa}^{N-1} C_{j,\kappa}[(\lambda_3((j+1)h) - \lambda_2((j+1)h))\beta_2^{\alpha} I^*((j+1)h) - \lambda_2((j+1)h)(\gamma_2^{\alpha} + \mu^{\alpha}) + \lambda_5((j+1)h)\beta_2^{\alpha}]$$

$$\lambda_3(\kappa h) = \lambda_3(T_f) - \sum_{j=\kappa}^{N-1} C_{j,\kappa}[1 + (\lambda_4((j+1)h) - \lambda_3((j+1)h))\varphi^{\alpha} - \lambda_3((j+1)h)\mu^{\alpha}]$$

$$\lambda_4(\kappa h) = \lambda_4(T_f) - \sum_{j=\kappa}^{N-1} C_{j,\kappa}[1 + (\lambda_3((j+1)h) - \lambda_1((j+1)h))\beta_1^{\alpha} S^*((j+1)h) + (\lambda_3((j+1)h) - \lambda_2((j+1)h))\beta_2^{\alpha} V^*((j+1)h) + (\lambda_5((j+1)h) - \lambda_4((j+1)h))u_2((j+1)h) - \lambda_4((j+1)h)\mu^{\alpha}]$$

$$\lambda_5(\kappa h) = \lambda_5(T_f) - \sum_{j=\kappa}^{N-1} C_{j,\kappa}[-\lambda_5((j+1)h)\mu^{\alpha}]$$

Step 8: Apply the optimal control law to compute $u_1(\kappa h)$ and $u_2(\kappa h)$ for all $\kappa = 1, 2, ..., N$ as follows:

$$u_1^*(\kappa h) = \min\{\max\{0, \frac{(\lambda_2(\kappa h) - \lambda_1(\kappa h))}{2A_1} S^*(\kappa h)\}, 1\}$$

$$u_2^*(\kappa h) = \min\{\max\{0, \frac{(\lambda_5(\kappa h) - \lambda_4(\kappa h))}{2A_2} I^*(\kappa h)\}, 1\}$$

Step 9: If the stopping criterion (the absolute value of optimal control of the current and the previous iterations) is held, then the algorithm ends, else return to Step 6.

The numerical simulation results show the changes in the behavior of the subpopulation when strategies 1 and 2 are applied. The number of subpopulations S, V, E and I decreases faster than without control i.e. when strategy 2 of treatment and vaccination is used. While we note that the recovered subpopulations R increase when strategies 1 and 2 are applied.

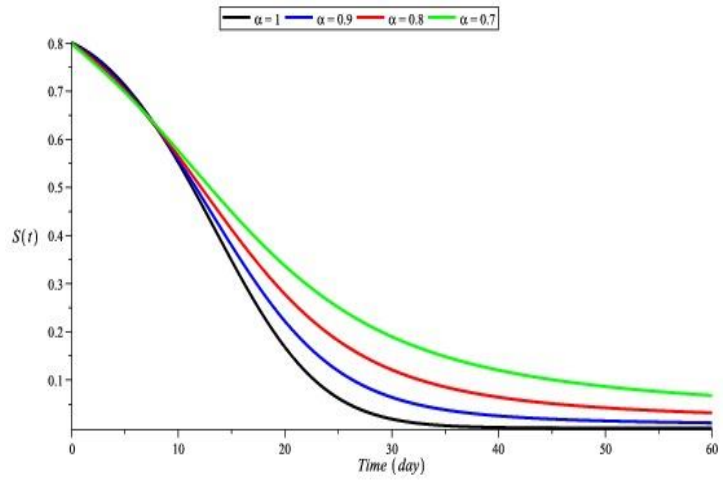

Figure 1: The Susceptible Subpopulation without u_1 and u_2.

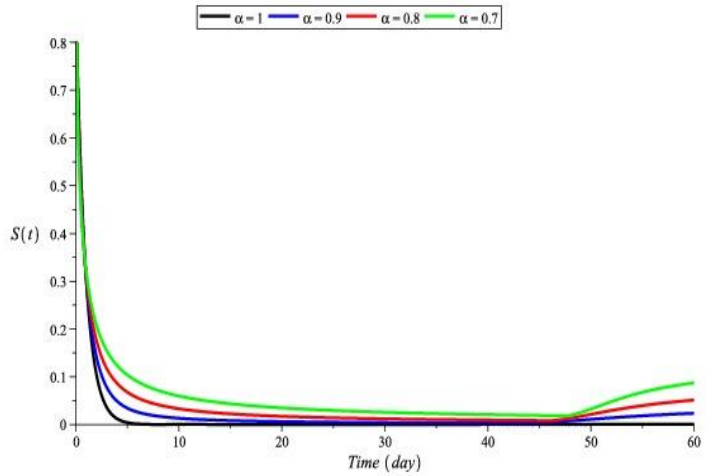

Figure 2: The Susceptible Subpopulation with u_1 and u_2.

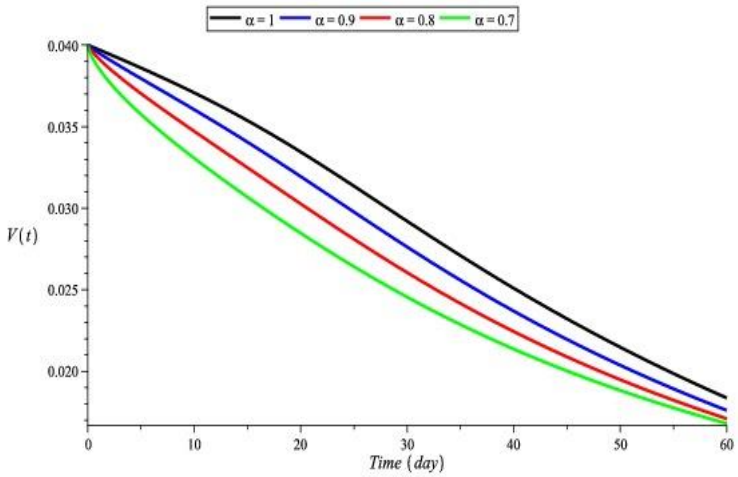

Figure 3: The Vaccinated Subpopulation without u_1 and u_2.

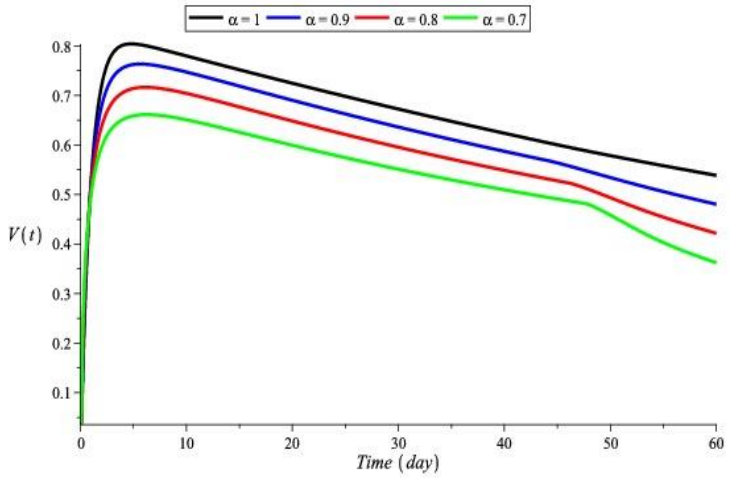

Figure 4: The Vaccinated Subpopulation with u_1 and u_2.

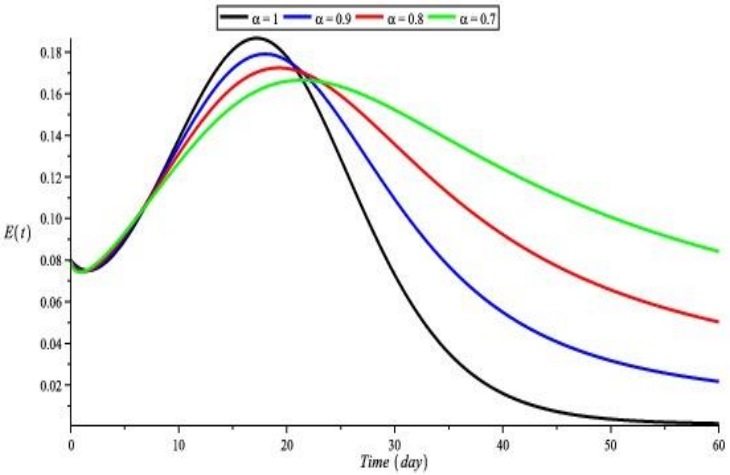

Figure 5: The Exposed Subpopulation without u_1 and u_2.

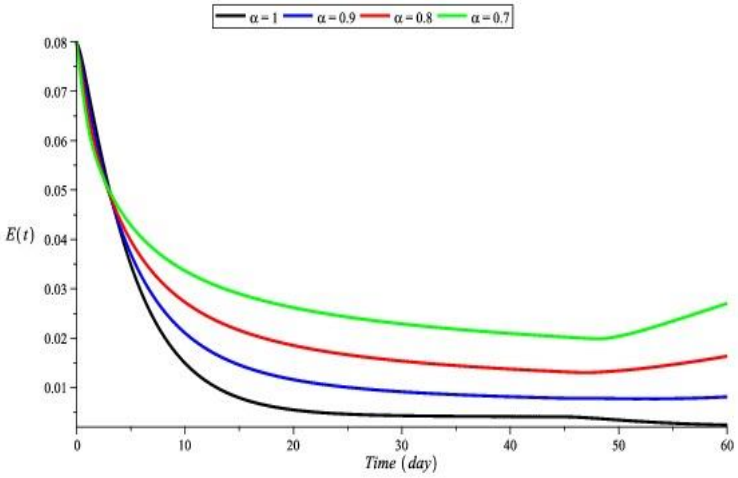

Figure 6: The Exposed Subpopulation with u_1 and u_2.

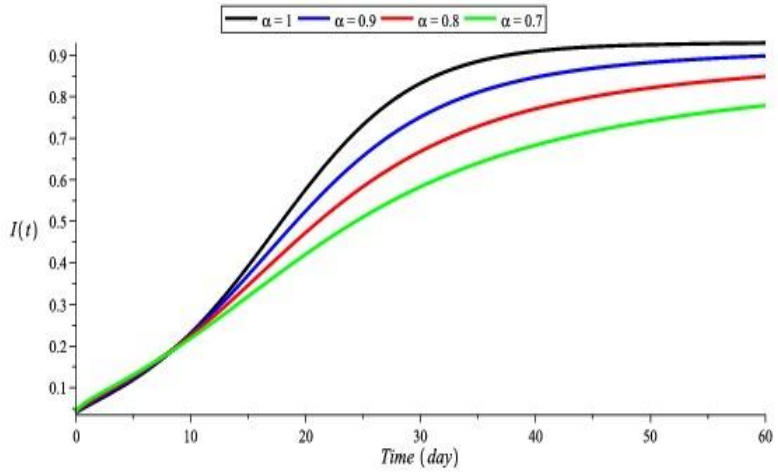

Figure 7: The Infected Subpopulation without u_1 and u_2.

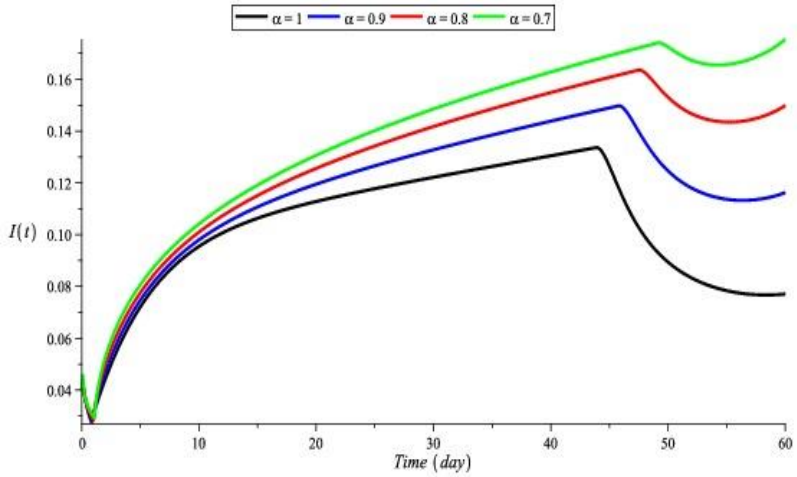

Figure 8: The Infected Subpopulation with u_1 and u_2.

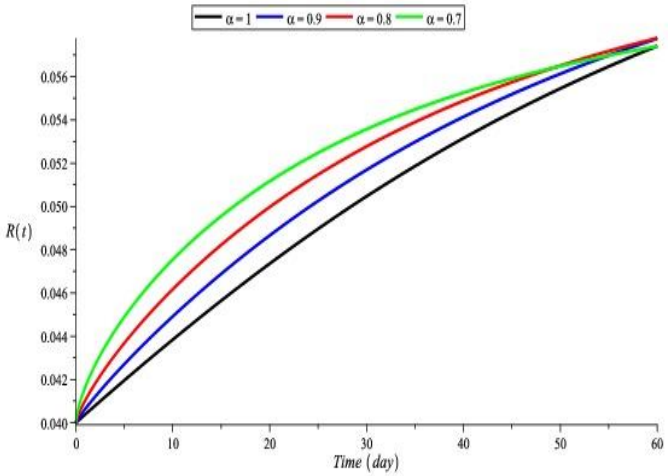

Figure 9: The Recovered Subpopulation without u_1 and u_2.

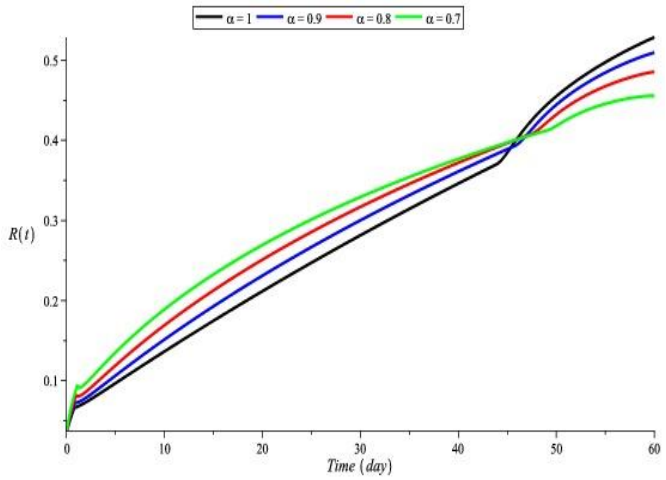

Figure 10: The Recovered Subpopulation with u_1 and u_2.

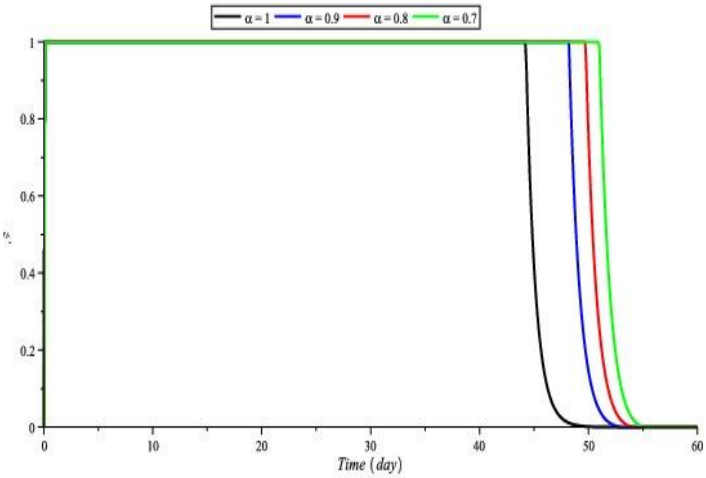

Figure 11: The Vaccination Control u_1.

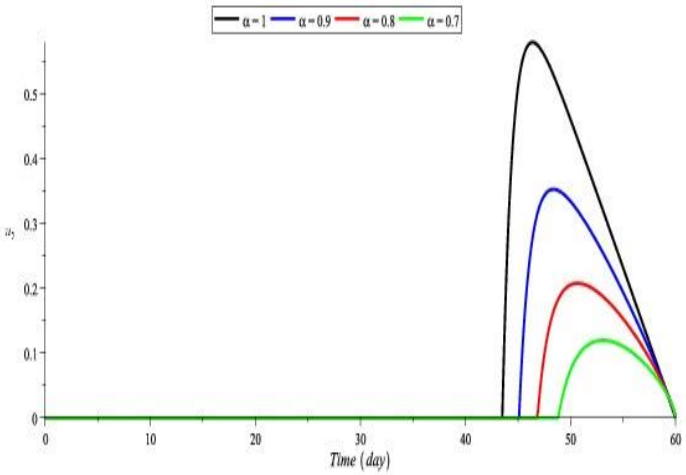

Figure 12: The Treatment Control u_2.

Conclusions

Mathematical modelling is a powerful tool for creating a comprehensive picture of disease spread behavior. In fact, the mathematical models described by the fractional derivative have more flexibility to describe more complicated real-life systems. The main goal of this work was to find optimal treatment and vaccination strategies to decrease the number of exposed and infected subpopulations so that the SARS disease doesn't spread too far and to reduce the cost of putting these strategies in place. The results of the study provided a novel treatment and vaccination strategy for halting the progression of *SARS* and preventing its further spread.

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السيطرة المثلى على مرض السارس من خلال النموذج الكسري

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المستخلص

المتلازمة التنفسية الحادة الوخيمة (سارس) مرض خطير للغاية يصيب الجهاز التنفسي للإنسان. في هذه المقالة ، نناقش كيفية السيطرة المثلى لهذا المرض من خلال نموذج وبائي SVEIR الجزئي مع متغيرين للتحكم (العلاج والتلقيح). لذا قمنا أولاً بتصميم مشكلة التحكم الأمثل الجزئي ثم تطبيق مبدأ Pontryagin الأدنى في صيغته الكسرية لايجاد اف الحل الامثل. أيضًا استخدمنا طريقة أويلر الكسرية الامامية والخلفية (FEM) لحل معادلات الحالة والحالة المرافقة ، على التوالي. أعطت النتائج استراتيجية علاج ولقاح جديدة لوقف ومنع انتشار مرض السارس.