# $p$-Graphs Associated with Some Groups and Vice Versa

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**ABSTRACT**

Given a finite group $G$ with a subset $X = \{x \neq e \in G / x^p = e\}$ where $p$ is a prime. The $p$-graphs denoted by $P_X(G)$ whose vertices are the elements of $G$ and where two distinct vertices $x$ and $y$ are adjacent if $x * y \in X$ or $y * x \in X$. The algebraic structure of such $p$-graphs has been investigated using Java programming language with NetBeans IDE. Then, a method is proposed to convert the graph structure to a group-like form.

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## 1. Introduction

Group theories and graph theories have been closely related for more than a century. To examine the theoretic algebraic properties of groups and vice versa, the robust combinatorial aspects of graphs have been used extensively. Beginning with Cayley graphs, which date back to 1878, graphs related to finite groups have a long history. Let $G$ be a nontrivial finite group and $S$ be a nonempty subset of $G$ that does not contain the identity element of $G$, the Cayley graph Cay(G , S) is a graph with vertex set $G$ and $(x, y) \in E(\Gamma)$ if and only if $x^{-1}y \in S$ for any $x, y \in G$ [1]. On the Cayley graph, many intriguing outcomes have been found see [2-6 ]. The intersection graph of a finite group G, denoted as $\beta$(G), is an undirected graph whose vertices are all nontrivial proper subgroups of G and whose two different vertices H and K are close when H ∩ K≠ 1 [7] . The independence graph of a finite groups is a graph whose vertices are the components G and where two vertices, x and y, are adjacent if there is a minimal generating set of G that includes x and y [8]. The graph with the group as its vertex set is known as a group's power graph. Two elements are considered to be adjacent if one of them is a power of the other. Keep in mind that while finite groups can be isomorphic or non-isomorphic, isomorphic power graphs on finite abelian groups require that the groups be isomorphic [9]. The power graph shows a number of fascinating results [10-12] .For graph theory has many researches in large fields and wide applications see [13-16]. This paper introduces $p$-graphs of finite groups with a subset $X$ such that whose vertices are the elements of $G$ and where two distinct vertices $x$ and $y$ are adjacent if $x * y \in X$ or $y * x \in X$. In addition, the algebraic and structure of $p$-graphs will be investigated, finally, the method of returning from the graphs to the groups will be discussed.

## 2. Preliminaries

A graph is a triple (V(Γ), E(Γ), $\psi$ ), where V(Γ) is a non-empty set called a vertex-set (which contains an element known as a vertex) and E(Γ) is a non-empty set called an edge-set (which contains an element known as an edge or link), and is the mapping of $E(\Gamma) \rightarrow V \times V$ . An edge's end vertices are referred to as incident with the edge, and vice versa. Two vertices are incident with the shared edge, and two edges are incident with the same vertex [17] .The graph's order and size are determined by the number of vertices $v = |V|$ and edges$\epsilon = |E|$ , respectively moreover, if $E = \emptyset$ , a graph is said to be empty [17]. The number of edges in a graph that intersect a vertex, or degree, is shown by the symbol $d_G(x)$ [17].

Any graph where $d_\Gamma\,(x) = \ d_\Gamma$ for any $x, y\ \in \Gamma(V)$ is said to be regular [7] . If two unique vertex pairs x and y exist in a graph, the path between them is defined as a sequence of vertex pairs $x = y_0, y_1, \dots , y_n = y$ such that $(y_i\,, y_{i+1}) \in E(\Gamma)$ for $0 \leq i \leq n-1$, and n is referred to as the length of this path [7] . If there is a path connecting any two vertexes, a graph is said to be connected [17]. A complete graph is one where every pair of vertices is connected by an edge, which is designated by $K_n$ [8]. In a graph $G$, a hamiltonian path is a path that passes through each vertex precisely once. A closed Hamiltonian path is known as a Hamiltonian cycle. If a graph has a Hamiltonian cycle, it is said to be Hamiltonian [18]. The diameter of a circle is equal to the maximum distance between any two vertices in a graph, and it is indicated by $dim(\Gamma)$ . The distance between any two vertices in a graph is equal to the number of edges in a shortest path between them denoted by $d_\Gamma(x, y)$ [17]. If an operation known as the product, which is represented by the symbol " $*$ " is defined in a nonempty set of elements $G$, then $G$ is said to constitute a group. such that $a\,, b \in G$ implies that $a * b \in G$ closed , and $a\,, b, c \in G$ Implies that $a * (b * c) = (a * b) * c$ associative low suggests that there is an element $e \in G$ such that $a * e = e * a = a$ foe all $a \in G$ , for any element an in G, there must exist an element $a \in G$ such that $a * a^{-1} = a^{-1} * a = e$ , indicating the presence of inverses in $G$ [19] .

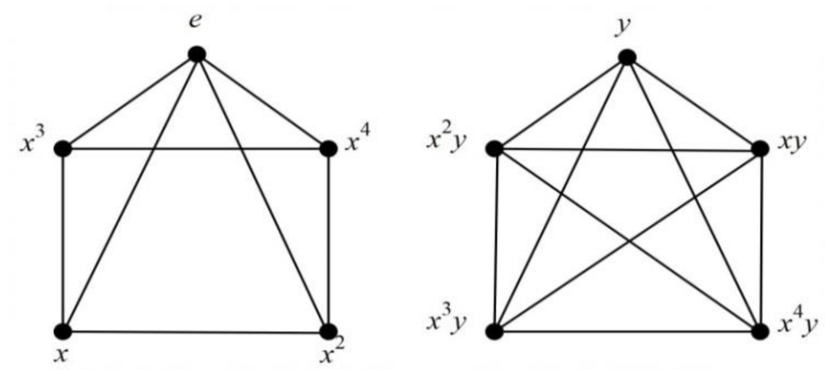
## 3. Main Result

**Definition 3.1** Let $(G\,,\ *\,)$ be a finite group and $X = \ \{x \in G\!: x \neq e\,,\ x^p = e\}$, $p$-graph $P_X\,(G)$ is defined such that whose vertices are the elements of $G$ and where two distinct vertices $x$ and $y$ are adjacent if $x * y \in X$ or $y * x \in X$

**Remark 3.2** It is worthwhile to observe the following:

1. $deg(e) = |X|$
2. $|X| = n - 1$ if $G = \ D_{2p}$ or $Z_p$ where $p$ is a prime

**Example 3.1.1** The graphs in figure 1 are the 5-graphs of the groups $(Z_5, +)$ and 3-graphs of the groups $(Z_6, +)$ with $X = \{1,2,3,4\}$ and $X = \{2,4\}$ respectively under the usual addition and the graph in figure 2 is 5-graphs of dihedral group of order 10 with $X = \{x\,, x^2, x^3, x^4\}$




$P_X(Z_5)$ $P_X(Z_6)$

Figure 1: $p$-graphs of groups integers module 5 and 6

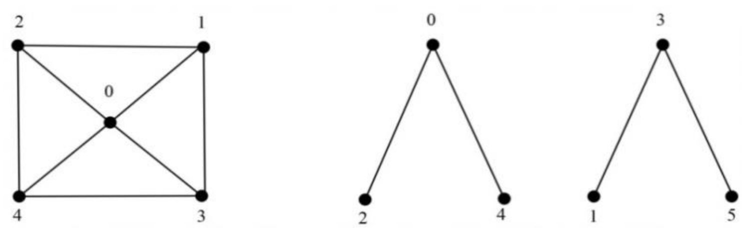



Figure 2: $p$ -graphs of dihedral group of order 10

**Lemma 3.3** All $p$-graphs of finite groups are not complete.

Proof. Suppose there exist an $p$-graph that is complete. let $x \neq y \in G$ such that $x = y^{-1}$ there is an edge between any two vertex, $x * y = e \in X$ , however, this is a contradiction of the identity element of $G$ not belonging to $X$ (see definition 3.1) so, there is no $p$-graph that is complete for any finite group $G$ .

**Lemma 3.4** For any finite group, $p$-graphs $P_X(G)$ is empty if and only if $|X| = 0$.

Proof. Suppose $P_X(G)$ is empty, $|X| = 0$ must be proved, since $P_X(G)$ is empty, there is no edge in $P_X(G)$ for any $x, y \in G$ with $x \neq y$, there is $x * y \notin X$ hence $X$ is empty, i.e $|X| = 0$ now suppose $|X| = 0$, for any $x, y \in G$ , $x * y \notin X$ that is mean there is no edge in $P_X(G)$ then $P_X(G)$ is empty.

**Theorem 3.5** Let $Z_p$ be a group of integer module $p$, where $p$ is a prime, then $P_X(Z_p)$ is connected graph and its diameter is 2.

Proof. By remark 3.2.2 the subset $X = n - 1$ where $|Z_p| = n$ , the identity element of $Z_p$ adjacent with all elements of $X$ By remark 3.2.1 so, there is a path between the identity element with all elements of $X$ where $|X| = n - 1$ contains all elements if $Z_p$ without the identity element, that is mean there is a path between any two vertex belong to $V\left(P_X(Z_p)\right)$ . The distance between $e$ and all the other vertices is one, the distance between any two vertex $x$ and $y$ is one such that $x \neq y^{-1}$ and the distance between any two vertex $x$ and $y$ is two such that $x = y^{-1}$ so, the diameter of $P_X(Z_p)$ is 2.

**Theorem 3.6** Let $Z_p$ be a group of integer module $p$, where $p$ is a prime, then $p$-graphs $P_X(Z_p)$ is Hamiltonian if $p \geq 5$.

Proof. $P_X(Z_p)$ is a connected graph, $X$ contains all elements of $Z_p$ without $e$ and $|X| = n - 1$ by remark 3.2.2 , since $deg(e) = |X|$ by remark 3.2.1 so, $deg(e) = n - 1 > \frac{n}{2}$ . From remark 3.2 it is known that for all vertex $x \in V\left(P_X(Z_p)\right)$, $x$ adjacent with all the other vertices excepted $x^{-1}$ since $x * x^{-1} = e \notin X$ and $P_X(G)$ has no loop by definition 3.1 So, $deg(x_i) = n - 2 \geq \frac{n}{2}$ So, by Dirac's theorem $P_X(Z_p)$ is Hamiltonian.

**Theorem 3.7** let $D_{2p}$ be a dihedral group of order $2p$, $p$ is a prime, then $P_X(D_{2p})$ is two components $H$ and $K$ one Component is complete and every component is a Hamiltonian graph.

Proof. Let $D_{2n} = \{e, x, x^2, \dots, x^{n-1}, y, xy, x^2y, \dots, x^{n-1}y\}$, $x^n = e$ , since $n = p$ so, $x^p = e$ that is mean $X = \{x, x^2, \dots, x^{n-1}\}$ there is an edge between any two vertex $x^i, x^j$ such that $x^i \neq {x^j}^{-1}$ Since $x^i * x^j = x^{i+j} \in X$ , so the component $H$ whose vertices $V(H) = \{x, x^2, \dots, x^{n-1}\}$ There is an edge between every vertex $x^i y, x^j y$ since $x^i y * x^j y = x^i y\, x^j y$ $= x^i x^{-j} y^2 = x^{i-j} \in X$ , So the component $K$ whose vertices $V(K) = \{y, xy, x^2y, \dots, x^{n-1}y\}\}$ So, the component $K$ is complete. $V(H) = \{x, x^2, \dots, x^{n-1}\}$ and $V(K) = \{y, xy, x^2y, \dots, x^{n-1}y\}\}$ and the subset $X = \{x, x^2, \dots, x^{n-1}\}$, $|X| = n - 1$ and from remark 3.2.1 $deg(e) = |X|$ so $deg(e) = n - 1 > \frac{n}{2}$ there is an edge between any two vertex $x^i$, $x^j$ such that $x^i \neq {x^j}^{-1}$ so, $deg(x^i) = n - 2 > \frac{n}{2}$ , now since $K$ is complete graph then $\varepsilon(K) = \frac{n^2 - n}{2} \geq \frac{n}{2}$ so by Dirac's theorem $H$ and $K$ are Hamiltonian graphs.

## 4. Induced groups by graphs theory

This section discusses new forms of graphs that have been generated from the group by using the methods that are mentioned previously.

### 4.1 The induced groups of a connected graph

Let $P_X(G)$ be a $p$-graph, where $G$ is a group whose elements coincide with the vertex-set $P_X(G)$. The identity element $e$ is the vertex that has a maximum degree in the connected $p$-graph, and the subset $X$ is the element adjacent to the identity element, and any two vertex non-adjacent that's means it inverse to the other. The closed property will be tested by Cayley's table for all the vertex in $P_X(G)$, if any two vertex $x$ and $y$ are adjacent then $x * y \in X$ but if the two vertex $x$ and $y$ in $P_X(G)$ are non-adjacent then $x * y \in G - X$ . In the forthcoming, several tables are established at which all possible possibilities are taken to get the table that achieves the closed property, and then the last property, which is associative law, will be tested.

**Example 4.1.1** The group by a graph is induced in figure 3.
$V\left(P_X(G)\right) = G$ , so $G = \{x, y, z\}$, take $x = e$ the identity element of $G$. So, $X = \{x, y\}$ then the two vertex $y$ and $z$ are non-adjacent that's mean $y^{-1} = z$ , now the closed property will be shown.

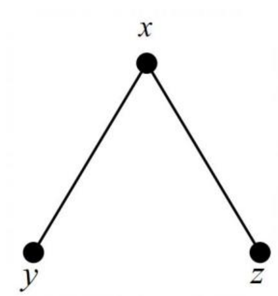



Figure 3: $p$-graph of order 3

Table 1: The associated law for $p$-graph of order 3

| $*$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ |
| $y$ | $y$ | | $x$ |
| $z$ | $z$ | $x$ | |

From the property of Cayley's table, $y * y = z$ and $z * z = y$ must be specified then Table 2 is obtained

Table 2: The associated law for $p$-graph of order 3

| $*$ | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ |
| $y$ | $y$ | $z$ | $x$ |
| $z$ | $z$ | $x$ | $y$ |

The associative law of the above table is checked by Java on NetBeans IDE so, $G = Z_3$ where $x = 0$ , $y = 1$ , $z = 2$ .

**Example 4.1.2** Group by a graph is induced in figure 4

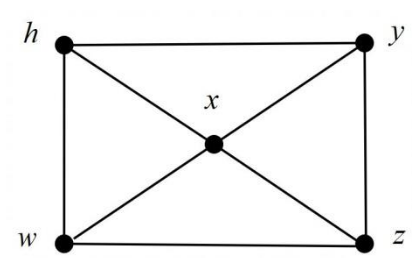



Figure 4: $p$-graph of order 5

$V(P_X(G)) = G$ , so $G = \{x, y, z, w, h\}$ since the vertex $x$ is the maximum degree in $P_X(G)$ take $x = e$ the identity element of $G$ . So, $X = \{y, z, w, h\}$ and $y^{-1} = w$ , $z^{-1} = h$ since two vertex

$y$ and $w$ are non-adjacent, $z$ and $h$ are non-adjacent. For the closed property, since there is an edge between $y$ and $z$ so, $y * z \in X = \{y, z, w, h\}$ but $y * z \neq y$ or $z$ since $y \neq e$ and $z \neq e$ so, $y * z = w$ or $h$, all probability is taken and filled with two different tables. This method is repeated for the other elements which are joined by an edge.

Table 3: The associated law for $p$-graph of order 5

| $*$ | $x$ | $y$ | $z$ | $w$ | $h$ |
|---|---|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ | $w$ | $h$ |
| $y$ | $y$ | $h$ | $w$ | $x$ | $z$ |
| $z$ | $z$ | $w$ | $y$ | $h$ | $x$ |
| $w$ | $w$ | $x$ | $h$ | $z$ | $y$ |
| $h$ | $h$ | $z$ | $x$ | $y$ | $w$ |

Table 4: The associated law for $p$-graph of order 5

| $*$ | $x$ | $y$ | $z$ | $w$ | $h$ |
|---|---|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ | $w$ | $h$ |
| $y$ | $y$ | $z$ | $h$ | $x$ | w |
| $z$ | $z$ | $h$ | $w$ | y | $x$ |
| $w$ | $w$ | $x$ | $y$ | $h$ | $z$ |
| $h$ | $h$ | $w$ | $x$ | $z$ | $y$ |

The associative law of the above two tables is checked by Java on Net Beans IDE So, the resulting group is $Z_5$ where $x = 0, y = 1, h = 2, z = 3, w = 4$ or $x = 0, y = 1, h = 3, z = 2, w = 4$.

## 5. Conclusions

This study has highlighted the graphs that can be generated by the groups. This paper discussed some properties of such graphs. The proposed work has shown that groups can be generated by graphs utilizing a new method with some examples.

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## حول الرسوم البيانية المتولدة من المجموعات المنتهية

نور الدين عدنان هاتو, علاء عامر نجم

قسم الرياضيات، كلية العلوم، جامعة البصرة

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