Approximation of bounded functions by positive linear operators in C_ρ

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Abstract

The aim of this paper is to define a new class of the operators defined by A. Lupaş for approximation continuous functions in the interval $[0, \infty)$. Our purpose is to study the convergence of this sequence of linear and positive operators and view some approximation properties which lead us to establish a Voronovskaja-type asymptotic formula for this operators. Finally, we study the rate of covrgence when we show this operators preserve properties of modulus of continuity on a continuous function.

Keywords: Korovkin theorem, Weighted space, Voronoviskaja-type formula, Modulus of continuity, Stirling number.

1.Introduction

At the International Dortmund Meeting held in the Witten (Germany- March-1995), A. Lupaş [1] studied the identity

$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k$, $|a| < 1$, where $(.)_k$ is the Pochammer's symbol defined by

$(\alpha)_0 = 1, \alpha \neq 0,$

$(\alpha)_k = (\alpha)(\alpha+1) \dots (\alpha+k-1), k \in \mathbb{N} := \{1,2,3, \dots\}.$

For more details, see [2], [3] and [4]. By putting $\alpha = nx$ and for $x \geq 0$ he introduced the sequence of linear positive operators

$L_n(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f(\frac{k}{n}),$

where the function $f:[0,\infty)\to\mathbb{R}$. After this, depending on the first condition of Bohman – Korovkin $L_n(1;x)=1$, Agratini in (1999) [5] found that $a={}^{1}/_{2}$ for the operators

$$L_n^*(f;x)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}\ f(\frac{k}{n})$$

He obtained the estimates for the order of the approximation on some finite interval $[0,b]$, $b>0$. Also he established a Voronoviskaja-type formula.

In (2007), A. Erençyn and F. Taşdelen [4], introduced a generalization of the operators L_n^* by the following operators

$$L_n^{**}(f;x)=2^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)_k}{2^k k!}\ f(\frac{k}{b_n}),$$

where $x\in\mathbb{R}_0:=[0,\infty)$, $n\in\mathbb{N}\coloneqq\{1,2,3,\dots\}$ and $\{a_n\},\{b_n\}$ are unbounded and increasing sequences of posotive numbers satisfying

$$\frac{a_n}{b_n}=1+O\left(\frac{1}{b_n}\right),\ \lim_{n\to\infty}\frac{1}{b_n}=0.$$

After that, in (2009), A. Erençyn and F. Taşdelen [6], estimated the rate of convergence of the Kantorovich – type version with the sequence of the operators L_n^{**}.

In (2012), Saddika Tarabie [7], studied the a- statistical convergence of two sequence of positive linear operators Λ_n, K_n one of them of discrete type and the other of integral type.

In (2014), Mohammad and Sadiq [2] defined a new sequence of linear positive operators $\tilde{L}_n$ defined by A. Lupaş as follow;

$$\tilde{L}_n(f;x)=\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ f\left(\frac{k+r}{n}\right), \tag{1}$$

where $G_x=\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}$ (2)

$$G_x:=\sum_{k=0}^{\infty}d_{n,k+r}(x)$$

So, we get

$$d_{n,r}(x)=2^{-nx}\frac{(nx)_r}{2^r\ r!} \tag{3}$$

In (2016), Haneen J. Sadiq [8] introduced a genralization of the operators $\tilde{L}_n$ given by (1) in two dimensions when she defined the operators $L_{n,m}^{\sim}$ with two variables (x,y) as follows

$$L_{n,m}^{\sim}(f;x,y)=\frac{1}{G_x}\frac{1}{G_y}\sum_{k=0}^{\infty}d_{n,k+r}(x)\sum_{j=0}^{\infty}d_{m,j+s}(y)\ \ f(\frac{k+r}{n},\frac{j+s}{m}),$$

We need some results to use them in this paper

2. Preliminaries

Theorem 1.1 (Korovkin Theorem) [8]:

For $x \in \mathbb{R}_0$, $f \in C_\rho$ and by applying Korovkin Theorem on the operators $\tilde{L}_n(f;x)$, we have:

1- $\tilde{L}_n(1;x) = 1$ (4)

2- $\tilde{L}_n(t;x) = x + \frac{2r}{n\ G_x} d_{n,r}(x).$ (5)

3- $\tilde{L_n}(t^2;x) = x^2 + \frac{2x}{n} + r d_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2 G_x}\right].$ (6)

4- $\tilde{L}_n(t^3;x) = x^3 + \frac{6x^2}{n} + \frac{6x}{n^2}$

$+\frac{8}{7n^3 G_x} r d_{n,r}(x)\left[r^2 + \frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right].$ (7)

5- $\tilde{L}_n(t^4;x) = x^4 + \frac{28x^3}{5\,n} + \frac{3511x^2}{105\,n^2} + \frac{158x}{7\,n^3}$

$+\frac{16}{15\,n^4\ G_x} r d_{n,r}(x)$

$\times\left[\begin{matrix}1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2\\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx\end{matrix}\right]$ (8)

Definition2.1 (m–th order moment): If $f \in C_\rho$ and for $x \in \mathbb{R}_0$, then $\tilde{T}_{n,m}(x) \equiv \tilde{L}_n((t-x)^m;x)$ is said to be the m –th order moment where $m \in \mathbb{N}_0 \coloneqq \{0,1,2,\ldots\}$.

Lemma 1.1 [8]: Let $r \in \mathbb{N}$, then for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, we have

1) $\tilde{T}_{n,0}(x) = 1.$

2) $\tilde{T}_{n,1}(x) = \frac{2r}{n\ G_x} d_{n,r}(x).$

3) $\tilde{T}_{n,2}(x) = r d_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2 G_x} + \frac{4x}{nG_x}\right) + \frac{2x}{n}.$

4) $\tilde{T}_{n,3}(x) = r d_{n,r}(x)\left(\frac{8r^2+2(nx+r)(3r+nx+5)+42+2nx(33+7nx)+8r(nx+3)}{7n^3 G_x} - \frac{2rx+6}{n^2 G_x}\right) + \frac{6x}{n^2}.$

5) $\tilde{T}_{n,4}(x) = rd_{n,r}(x)\Big[5.9428\frac{x^3}{nG_x} + 28.358\frac{x^2}{n^2G_x} + 36.9332\frac{x}{n^3G_x} + 4.8\frac{rx^2}{n^2G_x} + 0.3618\frac{rx}{n^3G_x} + + \frac{1.5237}{n^4G_x} + 1.0666\frac{r^3}{n^4G_x} + 1.7142\frac{r^2}{n^4G_x} - 7.9999\frac{r^2}{n^3G_x} + 12.3618\frac{r}{n^4G_x} + 0.5332\frac{xr^2}{n^3G_x} - 9.1428\frac{rx}{n^2G_x} - 9.1428\frac{x^2}{nG_x} - 9.4285\frac{x}{n^2G_x} - 19.4285\frac{r}{n^3G_x} - \frac{24}{n^3G_x}\Big] + \frac{12x^2}{n^2} + \frac{36x}{n^3}$.

So, here we deal with a new sequence of linear operators $\mathcal{M}_{n,s}$ as follow

$$\mathcal{M}_{n,s}(f(t);x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ f\left(x + \frac{t-x}{n^s}\right) \tag{9}$$

where $s > -\frac{1}{2}$ is a convenient approximation coefficient, $x \in \mathbb{R}_0$, $n \in \mathbb{N}$, $\mathbb{R}_0 = [0,\infty)$, and $\mathcal{M}_{n,s} \in P_n$, where P_n is the space of polynomials $P(x)$ of degree at most n , for all real numbers x. Note that this sequence of operators have a form very similar with Szãsz-Mirakyan operators.

Definition2.2 (Weighted space)[9]: The classes of functions which satisfying the condition $|f(x)| \le M_f\rho(x)$ with the norm $\|f\|_\rho = Sup_{x\epsilon[0,\infty)}\frac{|f(x)|}{\rho(x)}$, where $\rho(x) = 2^x$ are said to be weighted spaces, where $\rho(x)$ is a polynomial or exponential function such that continuous, monotonically increasing growths to infinity on $[0,\infty)$, such that $\rho(x) \ge 1$.
C_ρ: The subspace of all bounded continuous functions f

Remark:

In this paper we define a new operators (9) which represented a generalization of the previous operators $\tilde{L}_n$ defined in (1). Observe that the new operators $\mathcal{M}_{n,s}$ given in (9) when we put $s = 0$ we get the operators $L_n^{\sim}$ defined in (1) also its consequences. It mean that

$$\mathcal{M}_{n,0}(f(t);x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ f(t) = \tilde{L}_n(f(t);x)$$

3.Auxiliary results

$\mathcal{M}_{n,s}(f(t);x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ f\left(x + \frac{t-x}{n^s}\right)$

Firstly we introduce the next lemma

Lemma 3.1:

For $x \in [0,\infty)$, $m \in \mathbb{N}^0$, suppose that $\varphi_{n,m,r}(x) = \sum_{k=0}^{\infty} d_{n,k+r}(x) \ (k+r)^m$, where $d_{n,k+r}(x) = \ 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}$.

Then $x\varphi'_{n,m,r}(x) = \varphi_{n,m+1,r}(x) - 2\ \ln(2)^{nx}\ \varphi_{n,m,r}(x)$.

Proof: Applying the derivative

$$\frac{d}{dx}\varphi_{n,m,r}(x) = \sum_{k=0}^{\infty}(k+r)^m \frac{d}{dx}\{d_{n,k+r}(x)\}$$

$$= \sum_{k=0}^{\infty}(k+r)^m \left(\ 2^{-nx}\frac{d}{dx}\left\{\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\right\} - n\ln 2\left\{\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\right\}\right)$$

$x^{\bar{n}} = x(x+1)(x+2)\ldots(x+n-1) = \sum_{k+r} \begin{bmatrix} n \\ k+r \end{bmatrix} x^{k+r}$;

where $\begin{bmatrix} n \\ k+r \end{bmatrix} = S(n,k+r)$ is called Stirling number of the first kind [10].

Since $\frac{d}{dx}x^{\bar{n}} = \sum_{k+r} S(n,k+r)\ kx^{k-1} = \frac{d}{dx}(x)_{k+r}$ [10].

So, we need $nx^{\bar{n}} = (nx)_{k+r}$

$$= nx(nx+1)(nx+2)\ldots(nx+k+r-1) = \sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix}\ (nx)^{k+r} \equiv \sum_{k+r} S(n,k+r)\ (nx)^k.$$

Then $\frac{d}{dx}nx^{\bar{n}} = \sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix} \frac{d}{dx}(nx)^{k+r} = \sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix}\ kn(nx)^{k-1} = \frac{d}{dx}(nx)_{k+r}$,

So, $\frac{d}{dx}(nx)_{k+r} = n\sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix}\ k(nx)^{k-1}$.

$$\frac{d}{dx}\varphi_{n,m,r}(x) = \sum_{k=0}^{\infty}(k+r)^m\ 2^{-nx}\frac{n}{2^{k+r}(k+r)!}\sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix}\ k(nx)^{k-1}$$
$$- n\sum_{k+r=0}^{\infty}(k+r)^m\frac{(nx)_{k+r}}{2^{k+r}(k+r)!}\ln(2)$$

Since $(nx)_k = \sum_{k+r}\begin{bmatrix} n \\ k+r \end{bmatrix}\ k(nx)^{k+r}$

$$\frac{d}{dx}\varphi_{n,m,r}(x) = \ 2^{-nx}\frac{1}{x}\sum_{k=0}^{\infty}(k+r)^{m+1}\frac{(nx)_{k+r}}{2^{k+r}(k+r)!} - n\ln(2)\sum_{k=0}^{\infty}(k+r)^m\frac{(nx)_{k+r}}{2^{k+r}(k+r)!}$$

$$x\varphi'_{n,m,r}(x) = \varphi_{n,m+1,r}(x) - 2\ln(2)^{nx}\ \varphi_{n,m,r}(x). \quad \blacksquare$$

Lemma 3.2:

For $x \in [0,\infty)$, $m \in \mathbb{N}^0$, suppose that $\psi_{n,m,r}(x) = \sum_{k=0}^{\infty} d_{n,k+r}(x) \quad (x + \frac{t-x}{n^s})^m$, where $d_{n,k+r}(x) = \quad 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \ (k+r)!}$.

Then

$n^{2s+1}\psi_{n,m+1,r}(x) = xn^s\psi'_{n,m,r}(x) - mx(n^s - 1)\psi_{n,m-1,r}(x) - n^{s+1}(2^{nx}\ln(2)^x -$
$x(n^s - 1))\psi_{n,m,r}(x)$.

Proof: Clearly, $\psi_{n,m,r}(x) = \frac{1}{n^{ms}}\sum_{k=0}^{\infty} d_{n,k+r}(x) \quad (x(n^s - 1) + t)^m$,

It mean that $\psi_{n,m,r}(x) = \frac{1}{n^{ms}}\sum_{k=0}^{\infty} d_{n,k+r}(x) \quad (x(n^s - 1) + \frac{k+r}{n})^m$

Applying the derivative

$$\frac{d}{dx}\psi_{n,m,r}(x) = \frac{1}{n^{ms}}\sum_{k=0}^{\infty}\frac{d}{dx}\left\{d_{n,k+r}(x).(x(n^s-1)+\frac{k+r}{n})^m\right\}$$

$$= \frac{1}{n^{ms}}\sum_{k=0}^{\infty} d_{n,k+r}(x)m\left(x(n^s-1)+\frac{k+r}{n}\right)^{m-1}(n^s-1)$$

$$+\frac{1}{n^{ms}}\sum_{k=0}^{\infty}\left(x(n^s-1)+\frac{k+r}{n}\right)^m.\frac{d}{dx}d_{n,k+r}(x)$$

$$= \frac{1}{n^{ms}}\sum_{k=0}^{\infty} m(n^s-1)\left(x(n^s-1)+\frac{k+r}{n}\right)^{m-1} d_{n,k+r}(x)$$

$$+\frac{1}{n^{ms}}\sum_{k=0}^{\infty}\left(x(n^s-1)+\frac{k+r}{n}\right)^m\left\{\ 2^{-nx}\sum_{k=0}^{\infty}\frac{n}{2^{k+r}(k+r)!}\sum\nolimits_{k+r}\begin{bmatrix}h\\k+r\end{bmatrix}(k+r)(nx)^{k+r-1}\right.$$

$$\left.-n\ln(2)\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ \ (k+r)!}\right\}\psi'_{n,m,r}(x)$$

$$= \frac{m(n^s-1)n^{-s}}{n^{(m-1)s}}\sum_{k=0}^{\infty}\left(x(n^s-1)+\frac{k+r}{n}\right)^{m-1}\ 2^{-nx}\frac{(nx)_{k+r}}{2^{k+r}\ \ (k+r)!}$$

$$+\frac{1}{xn^{ms}}\sum_{k=0}^{\infty}(k+r)\ \left(x(n^s-1)+\frac{k+r}{n}\right)^m\ 2^{-nx}\frac{(nx)_{k+r}}{2^{k+r}\ \ (k+r)!}$$

$$-\frac{n\ \ln(2)}{2^{-nx}n^{ms}}\sum_{k=0}^{\infty}\left(x(n^s-1)+\frac{k+r}{n}\right)^m\ 2^{-nx}\frac{(nx)_{k+r}}{2^{k+r}\ \ (k+r)!}$$

$$\psi'_{n,m,r}(x) = \frac{m(n^s-1)n^{-s}}{n^{(m-1)s}} \sum_{k=0}^{\infty} \left(x(n^s-1) + \frac{k+r}{n}\right)^{m-1} d_{n,k+r}(x)$$

$$+ \frac{1}{xn^{ms}} \sum_{k=0}^{\infty} (k+r) \left(x(n^s-1) + \frac{k+r}{n}\right)^{m} d_{n,k+r}(x)$$

$$- \frac{n\ \ln(2)}{2^{-nx} n^{ms}} \sum_{k=0}^{\infty} \left(x(n^s-1) + \frac{k+r}{n}\right)^{m} d_{n,k+r}(x)$$

$\psi'_{n,m,r}(x) = \frac{m(n^s-1)}{n^s}\ \psi_{n,m-1,r}(x) + \sum_{k=0}^{\infty} \left\{\frac{(k+r)}{x} - n\ \ln(2) 2^{nx}\right\} \psi_{n,m,r}(x);$

$xn^s\psi'_{n,m,r}(x) = mx(n^s-1)\ \psi_{n,m-1,r}(x) + \sum_{k=0}^{\infty} n^s\{(k+r) - nx\ \ln(2)2^{nx}\}\psi_{n,m,r}(x).$

Suppose that I $\equiv \sum_{k=0}^{\infty} n^s\{(k+r) - nx\ \ln(2)2^{nx}\}\psi_{n,m,r}(x)$

$$\mathrm{I} = \frac{1}{n^{ms}} \sum_{k=0}^{\infty} d_{n,k+r}(x) \left(x(n^s-1) + \frac{k+r}{n}\right)^{m} . n^s\{(k+r) - nx\ \ln(2)2^{nx}\}$$

$$\mathrm{I} = \frac{n^{2s+1}}{n^{(m+1)s}} \sum_{k=0}^{\infty} d_{n,k+r}(x) \left(x(n^s-1) + \frac{k+r}{n}\right)^{m} . \left\{\frac{k+r}{n} + x(n^s-1) - x(n^s-1) + x\ \ln(2)2^{nx}\right\}$$

$$\mathrm{I} = \frac{n^{2s+1}}{n^{(m+1)s}} \sum_{k=0}^{\infty} d_{n,k+r}(x) \left(x(n^s-1) + \frac{k+r}{n}\right)^{m+1}$$

$$+ \frac{n^{2s+1}}{n^{(m+1)s}} \sum_{k=0}^{\infty} d_{n,k+r}(x) \left(x(n^s-1) + \frac{k+r}{n}\right)^{m} \left(x\ \ln(2)2^{nx} - x(n^s-1)\right)$$

$\mathrm{I} = n^{2s+1}\psi_{n,m+1,r}(x) + n^{s+1}\left(\ln(2)^x 2^{nx} - x(n^s-1)\right)\psi_{n,m,r}(x)$

Since $xn^s\psi'_{n,m,r}(x) = mx(n^s-1)\ \psi_{n,m-1,r}(x) + \mathrm{I}$

Then $xn^s\psi'_{n,m,r}(x) = mx(n^s-1)\ \psi_{n,m-1,r}(x) + n^{2s+1}\ \psi_{n,m+1,r}(x) + n^{s+1}\{\ln(2)^x\ 2^{nx} - x(n^s-1)\}\psi_{n,m,r}(x)$

Therefore we obtain;

$n^{2s+1}\ \psi_{n,m+1,r}(x) = xn^s\psi'_{n,m,r}(x) - mx(n^s-1)\ \psi_{n,m-1,r}(x) - n^{s+1}\{\ln(2)^x\ 2^{nx} - x(n^s-1)\}\psi_{n,m,r}(x).$ ∎

Theorem 3.1 (Bohman-Korovkin Theorem):

Let $f \in C_\rho$, $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $s > -\frac{1}{2}$, for the operators $\mathcal{M}_{n,s}$ given by (9) then;

1- $\mathcal{M}_{n,s}(1;x) = 1.$ (10)

2- $\mathcal{M}_{n,s}(t;x) = x + \frac{2r}{n^{s+1}G_x} d_{n,r}(x).$ (11)

3- $\mathcal{M}_{n,s}(t^2;x) = x^2 + \frac{2x}{n^{2s+1}} + r d_{n,r}(x) \left\{\frac{4x(n^s-1)}{n^{2s+1}G_x} + \frac{6nx+2r+6}{3n^{2s+2}G_x}\right\}.$ (12)

4- $\mathcal{M}_{n,s}(t^3;x) =$

$$x^3 + x^2 \left\{\frac{6}{n^{3s+1}}\left(\frac{r(n^s-1)^2}{G_x} d_{n,r}(x) + n^s\right)\right\} + x \left\{\frac{3}{n^{3s+2}}\left((n^s-1) r d_{n,r}(x) \left[\frac{6nx+2r+6}{3n^2 G_x}\right] + 2\right)\right\} + \frac{8}{7n^{3s+3}G_x} r d_{n,r}(x)$$

$$\times \left[r^2 + \frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]. \quad (13)$$

5- $\mathcal{M}_{n,s}(t^4;x) = x^4 + x^3 \left\{\frac{8(n^s-1)^3}{n^{4s+1}G_x} r d_{n,r}(x) + \frac{60(n^{2s}-1)+28}{5n^{4s+1}}\right\} + x^2 \left\{\frac{6(n^s-1)}{n^{4s+2}}\left[(n^s-1) r d_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2 G_x}\right) + 4\right] + \frac{3511}{105n^{4s+2}}\right\} + x \left\{\frac{158}{7n^{4s+3}} + \frac{32(n^s-1)}{7n^{4s+3}G_x} r d_{n,r}(x) \left[r^2 + \frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]\right\}$

$$+ \frac{16}{15\, n^{4s+4}\ G_x} r d_{n,r}(x)$$

$$\times \begin{bmatrix} 1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2 \\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx \end{bmatrix} \quad (14)$$

Proof: Clearly by accredition on results in [8] we get

1- $\mathcal{M}_{n,s}(1;x) = 1$, by applying Theorem 1.1 equation (4)

2- $\mathcal{M}_{n,s}(t;x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \left(x + \frac{t-x}{n^s}\right)$

$$= \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \frac{x(n^s-1)+t}{n^s} = \frac{1}{G_x}\frac{1}{n^s}\ 2^{-nx}\sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \{x(n^s-1)+t\}$$

$$= \frac{x(n^s-1)}{n^s}\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!} + \frac{1}{n^s}\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ t$$

$$= \frac{x(n^s-1)}{n^s}\tilde{L}_n(1;x) + \frac{1}{n^s}\tilde{L}_n(t;x)$$

Applying Theorem 1.1 equation (4) and (5), we get

$$= \frac{x(n^s-1)}{n^s} + \frac{1}{n^s}\{x + \frac{2r}{nG_x} d_{n,r}(x)\}$$

So, $\mathcal{M}_{n,s}(t;x) = x + \frac{2r}{n^{s+1}G_x} d_{n,r}(x)$

3- $\mathcal{M}_{n,s}(t^2;x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \left(x+\frac{t-x}{n^s}\right)^2$

$$= \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\left(\frac{x(n^s-1)+t}{n^s}\right)^2$$

$$= \frac{1}{G_x}\frac{1}{n^{2s}}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \{x^2(n^s-1)^2 + 2tx(n^s-1) + t^2\}$$

$$= \frac{x^2(n^s-1)^2}{n^{2s}}\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!} + \frac{2x(n^s-1)}{n^{2s}}\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ t$$

$$+\frac{1}{n^{2s}}\frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ t^2$$

$$= \frac{x^2(n^s-1)^2}{n^{2s}}\tilde{L}_n(1;x) + \frac{2x(n^s-1)}{n^{2s}}\tilde{L}_n(t;x) + \frac{1}{n^{2s}}\tilde{L}_n(t^2;x),$$

Directly, applying Theorem 1.1 equation (4), (5) and (6), we get

$$= \frac{x^2(n^s-1)^2}{n^{2s}} + \frac{2x(n^s-1)}{n^{2s}}\{x + \frac{2r}{nG_x}d_{n,r}(x)\} + \frac{1}{n^{2s}}\{x^2 + \frac{2x}{n} + rd_{n,r}(x)[\frac{6nx+2r+6}{3n^2G_x}]\}$$

$$= x^2\left\{\frac{(n^s-1)^2}{n^{2s}} + \frac{2(n^s-1)}{n^{2s}} + \frac{1}{n^{2s}}\right\} + \frac{2x}{n^{2s+1}} + rd_{n,r}(x)\left\{\frac{4x(n^s-1)}{n^{2s+1}G_x} + \frac{6nx+2r+6}{3n^{2s+2}G_x}\right\}$$

we obtained, $\mathcal{M}_{n,s}(t^2;x) = x^2 + \frac{2x}{n^{2s+1}} + rd_{n,r}(x)\left\{\frac{4x(n^s-1)}{n^{2s+1}G_x} + \frac{6nx+2r+6}{3n^{2s+2}G_x}\right\}$

4-$\mathcal{M}_{n,s}(t^3;x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \left(x+\frac{t-x}{n^s}\right)^3$

$$= \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\left(\frac{x(n^s-1)+t}{n^s}\right)^3$$

$$= \frac{1}{G_x} \frac{1}{n^{3s}} \; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \; (k+r)!} \; \{x^3(n^s-1)^3 + 3x^2(n^s-1)^2 t + 3x(n^s-1)t^2 + t^3\}$$

$$= \frac{x^3(n^s-1)^3}{n^{3s}} \frac{1}{G_x} \; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \; (k+r)!} \quad + \frac{3x^2(n^s-1)^2}{n^{3s}} \frac{1}{G_x} \; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \; (k+r)!} \; t$$
$$+ \frac{3x(n^s-1)}{n^{3s}} \frac{1}{G_x} \; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \; (k+r)!} \;\; t^2$$

$$+ \frac{1}{n^{3s}} \frac{1}{G_x} \; 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \; (k+r)!} \;\; t^3$$

$$= \frac{x^3(n^s-1)^3}{n^{3s}} \tilde{L}_n(1;x) + \frac{3x^2(n^s-1)^2}{n^{3s}} \tilde{L}_n(t;x) + \frac{3x(n^s-1)}{n^{3s}} \tilde{L}_n(t^2;x)$$

$$+ \frac{1}{n^{3s}} \tilde{L}_n(t^3;x)$$

Applying Theorem 1.1 equation (4), (5), (6) and (7), we get

$$= \frac{x^3(n^s-1)^3}{n^{3s}} + \frac{3x^2(n^s-1)^2}{n^{3s}} \left\{x + \frac{2r}{nG_x} d_{n,r}(x)\right\}$$
$$+ \frac{3x(n^s-1)}{n^{3s}} \left\{x^2 + \frac{2x}{n} + r d_{n,r}(x) \left[\frac{6nx+2r+6}{3n^2 G_x}\right]\right\}$$

$$+ \frac{1}{n^{3s}} \left\{x^3 + \frac{6x^2}{n} + \frac{6x}{n^2} \right.$$
$$+ \frac{8}{7n^{3s+3} G_x} r d_{n,r}(x) \left[r^2 \right.$$
$$\left.\left. + \frac{(nx+r)(3r+nx+5) + 21 + nx(33+7nx) + 4r(nx+3)}{4}\right]\right\}$$

$$= x^3 \left\{\frac{(n^s-1)^3}{n^{3s}} + \frac{3(n^s-1)^2}{n^{3s}} + \frac{3(n^s-1)}{n^{3s}} + \frac{1}{n^{3s}}\right\} + x^2\{\frac{6r(n^s-1)^2}{n^{3s+1} G_x} d_{n,r}(x) + \frac{6(n^s-1)}{n^{3s+1}}$$
$$+ \frac{6}{n^{3s+1}}\} + x\{\frac{3(n^s-1)}{n^{3s}} r d_{n,r}(x) \left[\frac{6nx+2r+6}{3n^2 G_x}\right] + \frac{6}{n^{3s+2}}\}$$
$$+ \frac{8}{7n^{3s+3} G_x} r d_{n,r}(x) \left[r^2 \right.$$
$$\left. + \frac{(nx+r)(3r+nx+5) + 21 + nx(33+7nx) + 4r(nx+3)}{4}\right]$$

$$= x^3 \left\{\frac{(n^s-1)^3 + 3(n^s-1)^2 + 3(n^s-1) + 1}{n^{3s}}\right\} + x^2\{\frac{6}{n^{3s+1}}\left(\frac{r(n^s-1)^2}{G_x} d_{n,r}(x) + n^s\right)\}$$

$$+ x\{\frac{3}{n^{3s+2}}\left((n^s-1) r d_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2 G_x}\right] + 2\right)\}$$

$$+ \frac{8}{7n^{3s+3}G_x} r d_{n,r}(x)\left[r^2 + \frac{(nx+r)(3r+nx+5) + 21 + nx(33+7nx) + 4r(nx+3)}{4}\right]$$

$$\mathcal{M}_{n,s}(t^3;x) = x^3 + x^2\{\frac{6}{n^{3s+1}}\left(\frac{r(n^s-1)^2}{G_x} d_{n,r}(x) + n^s\right)\}$$

$$+ x\{\frac{3}{n^{3s+2}}\left((n^s-1) r d_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2 G_x}\right] + 2\right)\}$$

$$+ \frac{8}{7n^{3s+3}G_x} r d_{n,r}(x)\left[r^2 + \frac{(nx+r)(3r+nx+5) + 21 + nx(33+7nx) + 4r(nx+3)}{4}\right]$$

$$\mathcal{M}_{n,s}(t^4;x) = \frac{1}{G_x}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \left(x + \frac{t-x}{n^s}\right)^4$$

$$= \frac{1}{G_x}\frac{1}{n^{4s}}\ 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_{k+r}}{2^{k+r}\ (k+r)!}\ \{x^4(n^s-1)^4 + 4x^3(n^s-1)^3 t + 6x^2(n^s-1)^2 t^2 + 4x(n^s-1)t^3 + t^4\}$$

$$= \frac{x^4(n^s-1)^4}{n^{4s}}\ \tilde{L}_n(1;x) + \frac{4x^3(n^s-1)^3}{n^{4s}}\ \tilde{L}_n(t;x) + \frac{6x^2(n^s-1)^2}{n^{4s}}\ \tilde{L}_n(t^2;x) + \frac{4x(n^s-1)}{n^{4s}}\ \tilde{L}_n(t^3;x) + \frac{1}{n^{4s}}\ \tilde{L}_n(t^4;x)$$

Applying Theorem 1.1 equation (4), (5), (6), (7) and (8), we get

$$= \frac{x^4(n^s-1)^4}{n^{4s}} + \frac{4x^3(n^s-1)^3}{n^{4s}}\left\{x+\frac{2r}{nG_x}d_{n,r}(x)\right\}$$
$$+\frac{6x^2(n^s-1)^2}{n^{4s}}\left\{x^2+\frac{2x}{n}+rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right]\right\}$$
$$+\frac{4x(n^s-1)}{n^{4s}}\left\{x^3+\frac{6x^2}{n}+\frac{6x}{n^2}\right.$$
$$\left.+\frac{8}{7n^3G_x}rd_{n,r}(x)\left[r^2+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]\right\}$$
$$+\frac{1}{n^{4s}}\left\{x^4+\frac{28x^3}{5\,n}+\frac{3511x^2}{105\,n^2}+\frac{158x}{7\,n^3}\right.$$
$$\left.+\frac{16}{15\,n^4\;G_x}rd_{n,r}(x)\begin{bmatrix}1.8214n^3x^3+15.3362n^2x^2+34.6249nx+1.4285+r^3+1.6071r^2\\+11.5892r+0.4999r^2nx+0.75rn^2x^2+5.9642rnx\end{bmatrix}\right\}$$

$$\mathcal{M}_{n,s}(t^4;x)=x^4\left\{\frac{(n^s-1)^4}{n^{4s}}+\frac{4(n^s-1)^3}{n^{4s}}+\frac{6(n^s-1)^2}{n^{4s}}+\frac{4(n^s-1)}{n^{4s}}+\frac{1}{n^{4s}}\right\}+x^3\left\{\frac{8(n^s-1)^3}{n^{4s+1}G_x}rd_{n,r}(x)+\right.$$
$$\left.\frac{12(n^s-1)^2}{n^{4s+1}}+\frac{24(n^s-1)}{n^{4s+1}}+\frac{28}{5n^{4s+1}}\right\}+x^2\left\{\frac{6(n^s-1)^2}{n^{4s+2}}rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right]+\frac{24(n^s-1)}{n^{4s+2}}+\frac{3511}{105n^{4s+2}}\right\}$$

$$+x\left\{\frac{158}{7n^{4s+3}}+\frac{32(n^s-1)}{7n^{4s+3}G_x}rd_{n,r}(x)\left[r^2\right.\right.$$
$$\left.\left.+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]\right\}$$

$$+\frac{16}{15\,n^{4s+4}\;G_x}rd_{n,r}(x)\begin{bmatrix}1.8214n^3x^3+15.3362n^2x^2+34.6249nx+1.4285+r^3+1.6071r^2\\+11.5892r+0.4999r^2nx+0.75rn^2x^2+5.9642rnx\end{bmatrix}$$

$$\mathcal{M}_{n,s}(t^4;x)=x^4+x^3\left\{\frac{8(n^s-1)^3}{n^{4s+1}G_x}rd_{n,r}(x)+\frac{60(n^{2s}-1)+28}{5n^{4s+1}}\right\}$$
$$+x^2\left\{\frac{6(n^s-1)}{n^{4s+2}}\left[(n^s-1)rd_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2G_x}\right)+4\right]+\frac{3511}{105n^{4s+2}}\right\}$$
$$+x\left\{\frac{158}{7n^{4s+3}}\right.$$
$$+\frac{32(n^s-1)}{7n^{4s+3}G_x}rd_{n,r}(x)\left[r^2\right.$$
$$\left.\left.+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]\right\}$$

$$+\frac{16}{15\,n^{4s+4}\;G_x}rd_{n,r}(x)$$

$$\times\begin{bmatrix}1.8214n^3x^3+15.3362n^2x^2+34.6249nx+1.4285+r^3+1.6071r^2\\+11.5892r+0.4999r^2nx+0.75rn^2x^2+5.9642rnx\end{bmatrix}. \qquad \blacksquare$$

The next lemma, we view the m–th order moment of the operators $\mathcal{M}_{n,s}$ as follows $\mathcal{T}_{n,s,m}(x)\equiv\mathcal{M}_{n,s}((t-x)^m;x)$ where $m\in\mathbb{N}_0:=\{0,1,2,\dots\}$. In this lemma we used Theorem 1.1 above to prove it.

Lemma 3.3 : Let $r\in\mathbb{N}$, $s>-\frac{1}{2}$ then for all $x\in\mathbb{R}_0$and $n\in\mathbb{N}$, we have

1- $\mathcal{T}_{n,s,0}(x)=1,$ (15)

2- $\mathcal{T}_{n,s,1}(x)=\frac{2}{n^{s+1}\,G_x}rd_{n,r}(x),$ (16)

3- $\mathcal{T}_{n,s,2}(x)=\frac{2x}{n^{2s+1}G_x}+rd_{n,r}(x)\left(\frac{4(n^s-1)}{n^{2s+1}G_x}+\frac{6nx+2r+6}{3n^{2s+2}G_x}-\frac{4x}{n^{s+1}G_x}\right),$ (17)

4- $\mathcal{T}_{n,s,3}(x)=\frac{6x}{n^{3s+2}}+rd_{n,r}(x)\left(\frac{(6x^2-12x+6)}{n^{s+1}G_x}+\frac{12x(1-x)}{n^{2s+1}G_x}+\frac{6x^2}{n3^{s+1}G_x}+\frac{6nx^2+2rx+6x}{n^{3s+2}G_x}-\frac{6nx^2+2rx+6x}{n^{2s+2}G_x}\right)+rd_{n,r}(x)\left[\frac{8r^2+2(nx+r)(3r+nx+5)+42+2nx(33+7nx)+8r(nx+3)}{7n^{3s+3}G_x}\right]$ (18)

5- $\mathcal{T}_{n,s,4}(x)=x^3\left\{\frac{60n^{2s}-32}{5n^{4s+1}}-\frac{24n^s}{n^{3s+1}}+\frac{12}{n^{2s+1}}\right\}+x^3\left\{\frac{8(n^s-1)^3}{n^{4s+1}}-\frac{24(n^s-1)^2}{n^{3s+1}}-\frac{8}{n^{s+1}}\right\}\frac{rd_{n,r}(x)}{G_x}+$

$$x^2\left\{\frac{2.520n^s+3508.48}{105n^{4s+2}}-\frac{24}{n^{3s+2}}\right\}$$

$$+x^2\left\{\frac{2(n^s-1)^2(6nx+2r+6)}{n^{4s+2}}-\frac{4(n^s-1)(6nx+2r+6)}{n^{3s+2}}+\frac{24(n^s-1)}{n^{2s+1}}+\frac{2(6nx+2r+6)}{n^{2s+2}}\right\}\frac{rd_{n,r}(x)}{G_x}$$

$$+x\left\{\frac{158}{7n^{4s+3}}-\frac{32n^{-s}rd_{n,r}(x)}{7n^{3s+3}G_x}\left[r^2+\frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]\right\}$$

$$+\frac{16}{15\,n^{4s+4}\,G_x}rd_{n,r}(x)$$

$$\times\begin{bmatrix}1.8214n^3x^3+15.3362n^2x^2+34.6249nx+1.4285+r^3+1.6071r^2\\+11.5892r+0.4999r^2nx+0.75rn^2x^2+5.9642rnx\end{bmatrix}. \qquad (19)$$

Next, we establish a Voronoviskaja-type formula.

Theorem 3.2 (Voronoviskaja Theorem):

Let $f \in C_\rho$, be twice differentiable and continuous at $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n^s \{\mathcal{M}_{n,s}(f;x) - f(x)\}$$
$$= f'(x)\left\{\frac{2r}{n^{s+1}\ G_x} d_{n,r}(x)\right\}$$
$$+ f''(x)\left\{\frac{x}{n^{s+1}} + r d_{n,r}(x)\left(\frac{2(n^s-1)}{n^{2s+1}G_x} + \frac{3nx+r+3}{n^{s+2}G_x} - \frac{2x}{nG_x}\right)\right\},$$

Proof: By Taylor's expansion of f about x;

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + (t-x)^2\varepsilon(t;x) \qquad \text{if } t \neq x,$$

where $\varepsilon(t;x) \to 0$ as $t \to x$

Substitute t by $\left(x + \frac{t-x}{n^s}\right)$,

$$f\left(x + \frac{t-x}{n^s}\right) = f(x) + \left(\frac{t-x}{n^s}\right)f'(x) + \left(\frac{t-x}{n^s}\right)^2\left[\frac{1}{2}f''(x) + \varepsilon\left(\frac{t-x}{n^s}\right)\right],$$

where ε is bounded function and $\lim_{h\to 0}\varepsilon(h) = 0$. Now by applying the operators $\mathcal{M}_{n,s}$ defined in (9), we get;

$$\mathcal{M}_{n,s}(f;x) = f(x) + \mathcal{T}_{n,s,1}(x)f'(x) + \frac{1}{2}\mathcal{T}_{n,s,2}(x)f''(x) + \mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^2\varepsilon\left(\frac{t-x}{n^s}\right);x\right).$$

$$\mathcal{M}_{n,s}(f;x) - f(x)$$
$$= \left\{\frac{2}{n^{s+1}\ G_x} r d_{n,r}(x)\right\} f'(x)$$
$$+ \left\{\frac{x}{n^{2s+1}G_x} + r d_{n,r}(x)\left(\frac{2(n^s-1)}{n^{2s+1}G_x} + \frac{3nx+r+3}{3n^{2s+2}G_x} - \frac{2x}{n^{s+1}G_x}\right)\right\} f''(x)$$
$$+ \mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^2\varepsilon\left(\frac{t-x}{n^s}\right);x\right)$$

applying Lemma 3.3 observe that $n^s\mathcal{T}_{n,s,1}(x) = 0$ as $n \to \infty$.

$$\lim_{n\to\infty} n^s\{\mathcal{M}_{n,s}(f;x)-f(x)\} = \left\{\frac{2}{n\ G_x} rd_{n,r}(x)\right\} f'(x) + \left\{\frac{x}{n^{s+1}G_x} + rd_{n,r}(x)\left(\frac{2(n^s-1)}{n^{s+1}G_x} + \frac{3nx+r+3}{3n^{s+2}G_x} - \frac{2x}{nG_x}\right)\right\} f''(x) + \lim_{n\to\infty} n^s \mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^2 \varepsilon\left(\frac{t-x}{n^s}\right); x\right)$$

Recalling Cauchy- Schwarz inequality and equation (17), we obtain

$$\left| n^s\mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^2 \varepsilon\left(\frac{t-x}{n^s}\right); x\right)\right| \le \left(\mathcal{M}_{n,s}\left(\varepsilon^2\left(\frac{t-x}{n^s}\right); x\right)\right)^{\frac{1}{2}} . \left(n^{2s}\mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^4; x\right)\right)^{\frac{1}{2}}.$$

From Theorem 3.1 and the properties of ε, we get:

$\lim_{n\to\infty} \mathcal{M}_{n,s}\left(\varepsilon\left(\frac{t-x}{n^s}\right); x\right) \equiv \lim_{n\to\infty}\mathcal{M}_{n,s}(\varepsilon(t); x) = \varepsilon(x) = 0$, then

$$\lim_{n\to\infty} \mathcal{M}_{n,s}\left(\varepsilon^2\left(\frac{t-x}{n^s}\right); x\right) \equiv \lim_{n\to\infty} \mathcal{M}_{n,s}(\varepsilon^2(t); x) = \varepsilon^2(x) = 0.$$

Now, from Lemma 3.3, we get:

$n^s\mathcal{M}_{n,s}\left(\left(\frac{t-x}{n^s}\right)^2 \varepsilon\left(\frac{t-x}{n^s}\right); x\right) \to 0$ as $n \to \infty$.

Therefore,

$\lim_{n\to\infty} n^s\{\mathcal{M}_{n,s}(f(t);x)-f(x)\} = \left\{\frac{2}{n\ G_x} rd_{n,r}(x)\right\} f'(x) + \left\{\frac{x}{n^{s+1}G_x} + rd_{n,r}(x)\left(\frac{2(n^s-1)}{n^{s+1}G_x} + \frac{3nx+r+3}{3n^{s+2}G_x} - \frac{2x}{nG_x}\right)\right\} f''(x).$ ■

4. Rate of Convergence

Definition 4.1[11] (Modulus of continuity):

Let $f \in C[a,b]$. For $\delta > 0$, then the modulus of continuity $\omega(f;\delta)$ is defined by

$\omega(f;\delta) = Sup_{|t-x|\le\delta}|f(t)-f(x)|$, for every $t, x \in [a,b]$, for $a, b \in \mathbb{R}$.

Next, for the order of approximation we give the following theorem:

Theorem 4.1[11]: Let $f \in C_\rho$, then $\left|\mathcal{M}_{n,s}(f;x)-f(x)\right| \le 2\omega(f;\delta)$,

where $\delta = \sqrt{\frac{2x}{n^{2s+1}G_x} + rd_{n,r}(x)\left(\frac{4(n^s-1)}{n^{2s+1}G_x} + \frac{6nx+2r+6}{3n^{2s+2}G_x} - \frac{4x}{n^{s+1}G_x}\right)}$.

Proof: By using the well-known property of modulus of continuity

$$|f(t)-f(x)| \leq \omega(f;\delta)\left(\frac{|t-x|}{\delta}+1\right)$$

Since the polynomials $\mathcal{M}_{n,s}(f;x)$ are linear positive operators, then we get

$$\left|\mathcal{M}_{n,s}(f;x)-f(x)\right| \leq \left(\frac{1}{\delta}\mathcal{M}_{n,s}(|t-x|;x)+1\right)\omega(f;\delta)$$

Applying Cauchy- Schwartz inequality, (10) and (17) we have

$$\left|\mathcal{M}_{n,s}(f;x)-f(x)\right| \leq \omega(f;\delta)\left(\frac{1}{\delta}\sqrt{\left(\mathcal{M}_{n,s}(t-x)^2;x\right)}+1\right)$$

$$\leq \omega(f;\delta)\left(\frac{1}{\delta}\left(\sqrt{\left|\mathcal{T}_{n,s,2}(x)\right|}\right)+1\right)$$

$$\leq \omega(f;\delta)\left(\frac{1}{\delta}\sqrt{\frac{2x}{n^{2s+1}G_x}+rd_{n,r}(x)\left(\frac{4(n^s-1)}{n^{2s+1}G_x}+\frac{6nx+2r+6}{3n^{2s+2}G_x}-\frac{4x}{n^{s+1}G_x}\right)}+1\right)$$

Choose $\delta=\sqrt{\frac{2x}{n^{2s+1}G_x}+rd_{n,r}(x)\left(\frac{4(n^s-1)}{n^{2s+1}G_x}+\frac{6nx+2r+6}{3n^{2s+2}G_x}-\frac{4x}{n^{s+1}G_x}\right)}$

Hence, we get

$$\left|\mathcal{M}_{n,s}(f;x)-f(x)\right| \leq 2\omega(f;\delta)$$ ■

Theorem 4.2: Let $f \in C_\rho$ on the interval $[0,\infty)$. Then for a real number $M>0$, the limit relation $\lim_{n\to\infty}\mathcal{M}_{n,s}(f;x)=f(x)$,

holds uniformly on the interval $[0,M]$.

Proof: By using (10), (11) and (12) from Theorem 3.1 we can see that:

$$\left\|\mathcal{M}_{n,s}(1;x)-1\right\|_{C[0,M]}=0$$

$$\left\|\mathcal{M}_{n,s}(t;x)-x\right\|_{C[0,M]}=\max_{x\in[0,M]}\frac{2r}{n^{s+1}G_x}d_{n,r}(x)$$

$\leq \frac{2r}{n^{s+1}G_M}d_{n,r}(M)\to 0$ as $n\to\infty$.

$$\left\|\mathcal{M}_{n,s}(t^2;x)-x^2\right\|_{C[0,M]}=\max_{x\in[0,M]}\left(\frac{2x}{n^{2s+1}}+rd_{n,r}(x)\left\{\frac{4(n^s-1)}{n^{2s+1}G_x}+\frac{6nx+2r+6}{3n^{2s+2}G_x}\right\}\right)$$

$\leq \frac{2M}{n^{2s+1}}+rd_{n,r}(M)\left\{\frac{4(n^s-1)}{n^{2s+1}G_M}+\frac{6nM+2r+6}{3n^{2s+2}G_M}\right\}\to 0$ for sufficiently large n.

The proof of Theorem 4.2 can be obtained by P. P. Korovkin [12]. ■

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تقريب دوال مقيدة بواسطة مؤثرات خطية موجبة في الفضاء C_ρ

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المستخلص

الهدف من هذا البحث هو تعريف فئة جديدة من مؤثرات معرفة بواسطة الكسندر لوباس لتقريب دوال مستمرة على الفترة $[0,\infty)$. غرضنا هو دراسة تقارب هذه المتتابعة من المؤثرات الخطية الموجبة وعرض بعض خصائص التقريب التي تقودنا لانشاء صيغة فرونوفسكي للتقارب. اخيراً درسنا معدل التقارب عندما وضحنا ان هذه المؤثرات تحافظ على خصائص معامل الاستمرارية على الدوال المستمرة.