T-Periodic Solution of Neutral Stochastic Functional Differential Equations with Infinite Delay

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Abstract

The aim of this work is to study the T-periodic solution of neutral type of stochastic functional differential equations with infinite delay (NSFDEwID), where we used the Lyapunov's second method to show the boundedness of the solution $x(t)$ and the solution map x_t to the above equations. Contraction mapping principle and Banach fixed point theorem are used in this work. We introduced an example in the end of this paper to illustrate the results of this work.

Keywords: T- Periodic, neutral stochastic differential equations, Lyapunov's second method.

1 Introduction

It is well known that the periodic phenomena have significant roles in nature, many systems behave periodically, for example, an average of repair or failed an item in a product, the wave vibration, the life cycle, environmental adjustments in four seasons, a satellite orbiting the Earth. Nevertheless, the challenge is how to get the periodic solutions of some periodic attitude after modelled via differential systems, whether the differential system is ordinary or stochastic. However, in the sensible case, the systems are often subject to stochastic perturbation. So, recently, the periodic solutions of SDEs have attracted great interests due to their applications in many ways. We refer the reader to [1-5] and references therein. Zhang and his colleagues [6] adopting the definitions 3.3 and 3.4 investigated the existence and uniqueness of stochastic periodic solutions to SDE in the form:

$$dx(t) = b(t; x(t))dt + \sigma(t; x(t))dw(t) \quad t \geq 0. \tag{2.1}$$

Hu and Xu [7] have investigated on the periodic stochastic Lotka-Voltra competitive-model with bounded delays and the periodic stochastic neural networks with infinite delay. They have generalized and improved the corresponding results in [4, 5,7, 8], where the existence theorems are generalized of M-valued periodic Markov process and M is a Polish space. Asker in [9, 10] studied Wellposedness and stability of neutral stochastic functional differential equations with infinite delay (NSFDEwID) in state space with the fading memory C_r.

The organization of this paper is as follow: preliminaries and proofs of required Lemmas by using Lyapunov's second method introduced in section 2. In section3 we study the T-periodic stochastic process by using contraction mapping principle and Banach Fixed Point Theorem.

In order to explain our results we introduce an example in section 4.

2 Perliminaries

Throughout this paper, unless otherwise specified, we use the following notation. R^d denotes the usual d-dimensional Euclidean space, $|\cdot|$ norm in R^d. If A is a vector or a matrix, its transpose is denoted by A^T; and $|A| = \sqrt{trace\ (A^T A)}$ its trace norm. Denote by $X^T Y$ the inner product of $X, Y \in R^d$. We choose the state space with the fading memory to be C_r defined as follows: for given positive number r,

$$C_r = \{\varphi \in C((-\infty, 0]; R^d) : \| \varphi \|_r = sup_{-\infty<\theta\leq 0} e^{r\theta} |\varphi(\theta)| < \infty\}, \quad (2.2)$$

where $C((-\infty, 0]; R^d)$ denotes the family of all bounded continuous R^d-value functions φ defined on $(-\infty, 0]$ to R^d with the norm $\| \varphi \|_r$. C_r is a Banach space with norm $\| \varphi \|_r = sup_{-\infty<\theta\leq 0} e^{r\theta} |\varphi(\theta)| < \infty$, see [15, 2], contains the Banach space of bounded and continuous functions and for any $0 < r_1 \leq r_2 < \infty, C_{r_1} \subset C_{r_2}$.

Let (Ω, F, P) be a complete probability space with a filtration $\{F_t\}_{t\in[0,+\infty)}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-null sets). Let K denote the family of all continuous increasing functions $\kappa: R_+ \to R_+$ such that $\kappa(0) = 0$ while $\kappa(u) > 0$ for $u > 0$. Let $K_\vee$ denote the family of all convex functions $\kappa \in K$ while $K_\wedge$ denote the family of all concave functions $\kappa \in K$ [11]. Let I_B denote the indicator function of a set B. $M^2([a, b]; R^d)$ is a family of process $\{x(t)\}_{a\leq t\leq b}$ in $L^2([a, b]; R^d)$ such that $E \int_a^b |x(t)|^2 dt < \infty$. The notation $P(C_r)$ denotes the family of all probability measures on $(C_r, B(C_r))$. Denote $C_b(C_r)$ the set of all bounded continuous functional.

For any $F \in C_b(C_r)$, $F: C_r \to R$ and $\pi(\cdot) \in P(C_r)$, let $\pi(F) := \int_{C_r} F(\phi)\pi(d\phi)$. M_0 stands for the set of probability measures on $(-\infty, 0]$, namely, for any $\mu \in M_0$, $\int_{-\infty}^0 \mu(d\theta) = 1.$ For any $r > 0$, let us further define M_r as follows, see [12]:

$$M_r := \{\mu \in M_0; \mu^{(r)} := \int_{-\infty}^0 e^{-r\theta} \mu(d\theta) < \infty\}. \quad (2.3)$$

Obviously, there exist many such probability measures and here we supply an example:

Example 2.1 let $\mu(d\theta) = e^{\beta\theta} d\theta$. Clearly, for any q < β,

$$\mu^{(q)} = \int_{-\infty}^0 e^{-q\theta} e^{\beta\theta} d\theta = \frac{1}{\beta-q} \int_{-\infty}^0 (\beta - q) e^{\theta(\beta-q)} d\theta = \frac{1}{\beta-q} < \infty, \quad (2.4)$$

Which implies $\mu^{(q)} \in M_q$ for any$q < \beta$.

Consider a d-dimensional neutral stochastic functional differential equations with infinite delay

$$d\{x(t) - D(t; x_t)\} = b(t; x_t)dt + \sigma(t; x_t)dw(t), x_0 = \xi = \{\xi(\theta): -\infty < \theta \leq 0\} \in C_r, \quad (2.5)$$

where

$$x_t = x(t + \theta): -\infty < \theta \leq 0$$

and $b, D: R \times C_r \to R^d$, $\sigma: R \times C_r \to R^{d\times m}$ are Borel measurable, F_t-adapted and there is some positive constant T such that $b(t + T; \phi) = b(t; \phi), \sigma(t + T; \phi) = \sigma(t; \phi) and\ D(t + T; \phi) = D(t; \phi)$and $\xi(t + T) = \xi(t)$ for any $t \in R$ and $\phi \in C_r$, i.e. b , σ, D and the initial data ξ are T-periodic in time t. $w(t)$ is an m-dimensional Brownian motion. Also, the coefficients $b(t; x_t)$, $\sigma(t; x_t)$ and the neutral term $D(t; x_t)$ of the system (2.5) satisfy the following assumptions:

(A1) For $\mu \in M_{2r}$ and $\varphi, \phi \in C_r$ there exist $k \in (0,1)$ with $\mu^{(2r)} < 1$ such that:

$$|D(t;\varphi) - D(t;\phi)|^2 \le k \int_{-\infty}^{0} |\varphi(\theta) - \phi(\theta)|^2 \mu(d\theta), \quad and \quad D(0;0) = 0 \qquad (2.6)$$

(A2) Let b be a continuous function. Assume there exist constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, and probability measure $\mu \in M_{2r}$ such that for any $\varphi, \phi \in C_r$

$$[\varphi(0) - \phi(0) - (D(t;\varphi) - D(t;\phi))]^T [b(t;\varphi) - b(t;\phi)] \le -\lambda_1 |\varphi(0) - \phi(0)|^2 + \lambda_2 \int_{-\infty}^{0} |\varphi(\theta) - \phi(\theta)|^2 \mu(d\theta), \qquad (2.7)$$

And for any function σ

$$|\sigma(t;\varphi) - \sigma(t;\phi)|^2 \le \lambda_3 |\varphi(0) - \phi(0)|^2 + \lambda_4 \int_{-\infty}^{0} |\varphi(\theta) - \phi(\theta)|^2 \mu(d\theta). \qquad (2.8)$$

Lemma 2.1 Assume that D, b and σ satisfy the conditions (2.6), (2.7) and (2.8) respectively, then there exists a unique global solution of the system (2.5).

Under Assumptions **(A1)** and **(A2)** , we observe that the system (2.5) has a unique global continuous solution $x(t)$ on $t > 0$ almost surely, which is continuous and F_t - adapted and can be express as follows:

$$x(t) = \xi(0) - D(0;\xi) + D(t;x_t) + \int_0^t b(t;x_s)ds + \int_0^t \sigma(t;x_s)dw(s). \qquad (2.9)$$

For the obvious benefit of Lyapunov's second method that does not need the knowledge of solutions of equations and thus has demonstrated great power in applications, we apply it here to prove the required lemmas. There are several references usable explain the main ideas of Lyapunov's second method for SDEs e.g, Khasminiskii [3], Mao [5], Kushner [12] and Arnold [14].

Now,

if $V \in C^{2,1}(R^d \times R_+; R_+)$, define the operator L such that

$$L[V(x(t) - D(t;x_t))] = V_t(x(t) - D(t;x_t)) + V_x(x(t) - D(t;x_t))b(t;x_t) + \frac{1}{2} trace[\sigma^T(t;x_t)V_{xx}(x(t) - D(t;x_t))\sigma(t;x_t)],$$

Where:

$$V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}), V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n\times n}, \quad i,j = 1, \dots, n.$$

The following lemma gives a criterion on the boundedness of 2-th moment for the solution.

Remark 2.2 [12, 9] Noting that for any positive $\lambda < 2r$, correspond to the definition of the norm $\| x_t \|_r^2$, it is easy to see that:

$$E \| x_t \|_r^2 = e^{-\lambda t} E \| \xi \|_r^2 + E(sup_{0<s\le t}|x(s)|^2). \qquad (2.10)$$

Lemma 2.3 For the system (2.4) let Assumptions **(A1)** and **(A2)** hold. If, in addition, there exist functions $V \in C^{2,1}(R^d \times R_+; R_+)$, $\kappa_1 \in K_\vee, \kappa_2 \in K_\wedge$ and positive numbers λ, β such that

$$\kappa_1(sup_{0<s\le t}|x(s) - D(s;x_s)|^2) \le V(x(s) - D(s;x_s)) \le \kappa_2(sup_{0<s\le t}|x(s) - D(s;x_s)|^2) \qquad (2.11)$$

and

$$LV(x(t) - D(t;x_t)) \le -\lambda V(x(t) - D(t;x_t)) + \beta \qquad (2.12)$$

for all $x(s) - D(s;x_s) \in C_r$. Then for any initial value $\xi \in C_r$, the 2-th moment of the solution $x(t)$ of equation (2.5) is bounded, say

$$E|x(t)|^2 \leq K \tag{2.13}$$

for all $t \geq 0$, where K is a positive constant. Moreover,

$$E \parallel x_t \parallel_r^2 \leq K. \tag{2.14}$$

Proof: For each integer k, define the stopping time

$$\tau_n = inf\{t \geq 0: \parallel x_t \parallel_r \geq n\} = inf\{t \geq 0: |x(t)| \geq n\},$$

it is clear that $\tau_n \uparrow \infty$ a.s. as $n \to \infty$. By Itô's formula, we have

$$E[e^{\lambda(\tau_n \wedge t)}| V(x(\tau_n \wedge t)) - D(\tau_n \wedge t; x_{\tau_n \wedge t})|^2] = E[V(x(0) - D(0; x_0))] + E\int_0^{\tau_n \wedge t} e^{\lambda s} LV(x(s) - D(s; x_s))ds + \lambda E\int_0^{\tau_n \wedge t} e^{\lambda s} V(x(s) - D(s; x_s)ds. \tag{2.15}$$

By (2.11) and (2.12), it follows that

$$E[e^{\lambda(\tau_n \wedge t)}\kappa_1(\left|x(\tau_n \wedge t) - D(\tau_n \wedge t; x_{\tau_n \wedge t})\right|^2)] \leq E[\kappa_2(|x(0) - D(0; x_0)|^2)] + E\int_0^{\tau_n \wedge t} e^{\lambda s}\beta ds.$$

If $n \to \infty$, then

$$E[e^{\lambda t}\kappa_1(|x(t) - D(t; x_t)|^2)] \leq E[\kappa_2(|x(0) - D(0; x_0)|^2)] + \frac{\beta}{\lambda}[e^{\lambda t} - 1],$$

Thus

$$E[\kappa_1(|x(t) - D(t; x_t)|^2)] \leq E[e^{-\lambda t}\kappa_2(|x(0) - D(0; x_0)|^2)] + \frac{\beta}{\lambda}[1 - e^{-\lambda t}].$$

Jensen's inequality yields to

$$\kappa_1(E[|x(t) - D(t; x_t)|^2]) \leq e^{-\lambda t}\kappa_2(E[|x(0) - D(0; x_0)|^2]) + \frac{\beta}{\lambda}.$$

Hence

$$E[|x(t) - D(t; x_t)|^2] \leq \kappa_1^{-1}(e^{-\lambda t}\kappa_2(E[|x(0) - D(0; x_0)|^2]) + \frac{\beta}{\lambda}). \tag{2.16}$$

By the assumption **(A1)**, the fact [(3.14), from [2]] and the equation (2.16), for any $\varepsilon > 0$, we have

$$sup_{0<s\leq t}(E[|x(s)|^2]) = sup_{0<s\leq t}(E[|x(s) - D(s; x_s) + D(s; x_s)|^2])$$
$$\leq [1 + \varepsilon]sup_{0<s\leq t}(E[|x(s) - D(s; x_s)|^2]) + [\frac{1}{\varepsilon} + 1]sup_{0<s\leq t}(E[|D(s; x_s)|^2])$$
$$\leq [1 + \varepsilon]\kappa_1^{-1}(e^{-\lambda t}\kappa_2(E[|x(0) - D(0; x_0)|^2]) + \frac{\beta}{\lambda}) + k[\frac{1}{\varepsilon} + 1][e^{-\lambda s}\mu^{(\lambda)}E \parallel \xi \parallel_r^2 + sup_{0<s\leq t}(E|x(s)|^2)],$$

Take $\varepsilon > \frac{k}{1-k}$ implies $\gamma = k(1 + \frac{1}{\varepsilon}) < 1$, we arrive at

$$sup_{0<s\leq t}(E|\, x(s)|^2) \leq \frac{1+\varepsilon}{1-\gamma}\kappa_1^{-1}(e^{-\lambda t}\kappa_2(E[|x(0) - D(0; x_0)|^2]) + \frac{\beta}{\lambda}) + \frac{\gamma e^{-\lambda s}\mu^{(\lambda)}}{1-\gamma}E \parallel \xi \parallel_r^p , \tag{2.17}$$

Hence,

$$limsup_{t\to\infty}(E|\, x(t)|^2) \leq \frac{(1+\varepsilon)}{(1-\gamma)}\kappa_1^{-1}(\frac{\beta}{\lambda}). \tag{2.18}$$

Thus, there exist a $S > 0$ such that $E|x(s)|^2 \leq \frac{1.5(1+\varepsilon)}{1-\gamma}\kappa_1^{-1}(\frac{\beta}{\lambda})$ for all $t \geq S$. Also, because of continuity of $|x(s)|^2$, there is a $K_0 > 0$ such that $|x(s)|^2 \leq K_0$ for $t \geq S$.

Let $K = max\{\frac{1.5(1+\varepsilon)}{(1-\gamma)}\kappa_1^{-1}(\frac{\beta}{\lambda}), K_0\}$, this mean we have for all $t \geq 0$, $E|x(s)|^2 \leq K$. Moreover, since

$$E \parallel x_t \parallel_r^2 \leq e^{-\lambda t} \parallel \xi \parallel_r^2 + E(sup_{0<s\leq t}|x(s)|^2), \quad (2.19)$$

Thus,

$$limsup_{t\to\infty}(E \parallel x_t \parallel_r^2) \leq K.$$

Now we consider the difference between two solutions of (2.4) starting from different initial data, that is

$$d\Big(x(t;\xi) - x(t;\eta) - D\big(t;x_t(\xi)\big) + D\big(t;x_t(\eta)\big)\Big) = \{b\big(t;x_t(\xi)\big) - b\big(t;x_t(\eta)\big)\}dt + \{\sigma(t;x_t(\xi)) - \sigma(t;x_t(\eta))\}dw(t), \quad (2.20)$$

Where, $x(t;\xi)$ and $x(t;\eta)$ two different solutions with two different initial data ξ,η to the system (2.5). The following lemma will show that $E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2$ is uniformly continuous on $[0,\infty)$, which will be used later. And the idea for our proof comes from [6,9,11].

Lemma 2.4 Suppose all the conditions of Lemma 2.3 hold and $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$ *and* $\lambda \in (0, \frac{1}{M}[2\lambda_1 - 73\lambda_3 - 2\lambda_2\mu^{(2r)} - 73\lambda_4\mu^{(2r)}] \wedge 2r)$ where $M = (1+k)(1+\mu^{(2r)})$. *Then* $E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2$ are uniformly continuous on the entire $t \in [0,\infty)$. *Moreover*, $\lim_{t\to\infty} E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2 = 0$.

Proof: By [Lemma 4.4 [2]], $E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2 \leq C_6 E \parallel \xi - \eta \parallel_r^2 e^{-\lambda t}$ where C_6 is a constant dependent of only λ, k and $\mu^{(2r)}$.

This implies that $E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2$ is uniformly continuous on the entire [0, ∞] and $\lim_{t\to\infty} E \parallel x_t(\xi) - x_t(\eta) \parallel_r^2 = 0$.

For a given function $U \in C^{2,1}(R \times R^d \times R_+; R_+)$ and any two solutions of (2.4) $x(t), y(t)$ where $t \geq 0$, we define an operator $LU: R \times R^d \times R^d \to R$ associated with the equation (2.18) by

$$L\,U(x(t) - D(t;x_t), y(t) - D(t;y_t)) = U_t(x(t) - D(t;x_t) - (y(t) - D(t;y_t)) + U_x(x(t) - D(t;x_t) - (y(t) - D(t;y_t))[b(t;x_t) - b(t;y_t)] + \frac{1}{2}trace[(\sigma(t;x_t) - \sigma(t;y_t))^T U_{xx}(x(t) - D(t;x_t) - (y(t) - D(t;y_t))(\sigma(t;x_t) - \sigma(t;y_t))].$$

Lemma 2.5 Let the conditions of the Lemma 2.3 hold. Assume further that there are functions $U \in C^{2,1}(R \times R^d \times R_+), \kappa_3 \in K_\wedge$ *and* $\kappa_4 \in K_\vee$ *such that*

$$U(x(t) - D(t;x_t)) \leq \kappa_3(|x(t) - D(t;x_t)|^2) \quad for \quad all \quad x(t) \in C_r \quad (2.20)$$

And

$$LU(x(t) - D(t;x_t), y(t) - D(t;y_t)) \leq -\kappa_4(|x(t) - D(t;x_t) - (y(t) - D(t;y_t))|^2), \quad (2.21)$$

for all $x(t), y(t) \in C_r$.

If initial values ξ and η for the solutions x and y, respectively, are in C_r, then

$$\lim_{t\to\infty} E|x(t) - y(t) - (D(t;x_t) - D(t;y_t)|^2 = 0 \quad (2.22)$$

Proof: For any positive number n, define

$$\alpha_n = inf\{t \geq 0: |x(t) - y(t)| \geq n\} = inf\{t \geq 0: \parallel x_t - y_t \parallel_r \geq n\}.$$

It is clear that $\alpha_n \to \infty$, when $n \to \infty$.

Set $t_n = \alpha_n \wedge t$ and $\Gamma_{x;y}(t_n) = x(t_n) - y(t_n) - (D(t_n, x_{t_n}) - D(t_n; y_{t_n}))$, by applying Itô's formula to $U(\Gamma_{x;y}(t_n))$ yields

$$EU(\Gamma_{x;y}(t_n)) = EU(\xi - \eta - (D(t_n;\xi) - D(t_n;\eta)) + E\int_0^{t_n} LU(\Gamma_{x;y}(s))ds.$$

So by conditions (2.20), (2.21) and then letting $n \to \infty$, we have

$$0 \le E(\kappa_3(|\xi - \eta - (D(t;\xi) - D(t;\eta))|^2) - E\int_0^t \kappa_4(|\Gamma_{x;y}(s)|^2)ds,$$

Thus

$$E\int_0^t \kappa_4(|\Gamma_{x;y}(s)|^2)ds \le E(\kappa_3(|\xi - \eta - (D(t;\xi) - D(t;\eta))|^2).$$

Using Jensen's inequality results in

$$\int_0^t \kappa_4(E(|\Gamma_{x;y}(s)|^2)ds \le \kappa_3(E(|\xi - \eta - (D(t;\xi) - D(t;\eta))|^2) < \infty. \tag{2.23}$$

Now we claim $\lim_{t\to\infty} E|\Gamma_{x;y}(t)|^2 = 0$, If this assertion is not true, then there is some $\varepsilon > 0$ and a sequence $\{t_n\}_{n\ge 1}$ satisfying $0 \le t_n \le t_n + 1 \le t_{n+1}$ such that

$$\lim_{t\to\infty} E|\Gamma_{x;y}(t)|^2 \ge \varepsilon, \quad n \ge 1.$$

By Lemma 2.4 , there is a positive constant C such that:

$$|E|\Gamma_{x;y}(t)|^2 - E|\Gamma_{x;y}(s)|^2| \le C.$$

Let $\delta = 1 \wedge (\varepsilon/2C)$, then, for $t_n \le s \le t_n + \delta$, we can get

$$E|\Gamma_{x;y}(s)|^2 \ge E|\Gamma_{x;y}(t_n)|^2 - E|E|\Gamma_{x;y}(s)|^2 - E|\Gamma_{x;y}(t_n))|^2| \ge \varepsilon - C \ge \varepsilon - C\delta \ge \frac{\varepsilon}{2}.$$

Consequently

$$\int_0^\infty \kappa_4(E(|\Gamma_{x;y}(s)|^2)ds \ge \sum_{n=1}^\infty \int_{t_n}^{t_n+\delta} \kappa_4(E(|\Gamma_{x;y}(s)|^2)ds$$
$$\ge \sum_{n=1}^\infty \int_{t_n}^{t_n+\delta} \kappa_4(\tfrac{\varepsilon}{2})ds \tag{2.24}$$

But this is in contradiction with (2.23). So

$$\lim_{t\to\infty} E|\Gamma_{x;y}(t)|^2 = 0, \tag{2.25}$$

3 T-periodic stochastic process

In this section, we present and prove our main theorem. The main technique that we use in this part is based on contraction mapping principle and Banach Fixed Point Theorem (Lyapunovs second method).

Definition 3.1 Let (X, d) be a metric space. Then a map $f: X \to X$ is called a contraction mapping on X if there exists $q \in [0,1)$ such that $d(f(x), f(y)) \le q d(x, y)$ for all x, y in X.

Theorem 3.2 Banach Fixed Point Theorem. Let (X, d) be a non-empty complete metric space with a contraction mapping $f: X \to X$. Then f admits a unique fixed-point x^* in X (i.e. $f(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = f(x_{n-1})$, then $x_n \to x^*$.

Now, we state the definition of periodic stochastic process and stochastic periodic solution.

Definition 3.3 [6] A stochastic process $x(t), t \ge 0$ is said to be a T-periodic stochastic process, if the stochastic processes $y(t) := x(t + T), t \ge 0$ and $x(t), t \ge 0$ have the same finite-dimensional distributions.

Definition 3.4 [6] If $\mathrm{x(t)}, \mathrm{t} \geq 0$ is a solution of (2.5) and $\mathrm{x(t)}$ is a T-periodic stochastic process, then $\mathrm{x(t)}$ is said to be a stochastic periodic solution with period T of (2.5).

Theorem 3.5 Assume that the conditions of Lemmas 2.3 and 2.5 are all satisfied, then (2.5) admits a unique T-periodic stochastic periodic solution.

Proof: For an arbitrary $\xi;\eta \in L^2(\Omega; C_r)$, define a metric $d(\xi,\eta) = (E\|\xi-\eta\|_r^2)^{\frac{1}{2}}$, then $L^2(\Omega; C_r)$ is a complete metric space. From Lemma 2.3, we get that for any $t \in [0,\infty)$, the solutions $x(t;\xi); x(t;\eta) \in L^2(\Omega; C_r)$.

Define a mapping $f: L^2(\Omega; C_r) \to L^2(\Omega; C_r)$ by $f(\xi) = x(T;\xi)$, there is a constant $M > 0$ such that for any integer $m > M$, the mapping $f^m(\xi) = f(\xi) \circ \cdots \circ f(\xi) = x(mT,\xi)$, the mapping $f(\xi)$ composed with itself m times, then by Lemma 2.5, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} d^2(f^m(\xi); f^m(\eta)) &= d^2\big(x(mT;\xi); x(mT;\eta)\big) \\ &= E|x(mT;\xi) - x(mT;\eta)|^2 \\ &< \varepsilon^2 E \parallel \xi - \eta \parallel_r^2 = \varepsilon^2 d^2(\xi,\eta), \end{aligned}$$

That is, by define the metric d,

$$d(f^m(\xi); f^m(\eta)) < \varepsilon d(\xi,\eta),$$

Therefore, f is a contraction mapping on the complete metric space $L^2(\Omega; C_r)$, and so there exists a unique fixed point $\gamma \in L^2(\Omega; C_r)$ such that $\gamma = f\gamma = x(T,\gamma)$.

Now we are in the position to prove that $x(t;\gamma)$ is the unique T-periodic stochastic periodic solution of (2.5).

Under Assumptions **(A1)** and **(A2)**, $x(t;\gamma)$ satisfy the following NSFDEwID:

$$\begin{aligned} x(t;\gamma) = \gamma(0) - D(0;\gamma) + D(t; x_t(\gamma)) + \int_0^t b(s; x_s(\gamma))ds \\ + \int_0^t \sigma(s; x_s(\gamma))dw(s), \quad t \geq 0. \end{aligned} \tag{3.1}$$

Let $t = T$ and $t = t + T$ in (3.1) respectively, we get

$$\begin{aligned} x(T;\gamma) = \gamma(0) - D(0;\gamma) + D(T; x_{t+T}(\gamma)) + \int_0^T b(s; x_s(\gamma))ds \\ + \int_0^T \sigma(s; x_s(\gamma))dw(s), \end{aligned} \tag{3.2}$$

$$\begin{aligned} x(t+T;\gamma) = \gamma(0) - D(0;\gamma) + D(t+T; x_{t+T}(\gamma)) + \int_0^{t+T} b(s; x_s(\gamma))ds \\ + \int_0^{t+T} \sigma(s; x_s(\gamma))dw(s). \end{aligned} \tag{3.3}$$

Consequently from (3.2) and (3.3), we have

$$\begin{aligned} x(t+T;\gamma) &= x(T;\gamma) - D(T; x_{t+T}(\gamma)) + D(t+T; x_{t+T}(\gamma)) \\ &\quad + \int_T^{t+T} b(s; x_s(\gamma))ds + \int_T^{t+T} \sigma(s; x_s(\gamma))dw(s). \end{aligned} \tag{3.4}$$

Let $s = r + T$, $\widetilde{w}_t = w_{t+T} - w_t$, the probability space $(\Omega; F; P)$ is fixed, then by the (3.4) we have:

$$\begin{aligned} x(t+T;\gamma) &= x(T;\gamma) - D(T; x_T(\gamma)) + D(t+T; x_{t+T}(\gamma)) \\ &\quad + \int_0^t b(r+T; x_{r+T}(\gamma))dr + \int_0^t \sigma(r+T; x_{r+T}(\gamma))d\widetilde{w}(r). \end{aligned}$$

Note that

$$b(t+T; x_{t+T}(\gamma)) = b(t; x_{t+T}(\gamma)), \quad \sigma(t+T; x_{t+T}(\gamma)) = \sigma(t; x_{t+T}(\gamma)),$$

Therefore for $t \geq 0$,

$$x(t+T;\gamma) = x(T;\gamma) - D(T; x_T(\gamma)) + D(t+T; x_{t+T}(\gamma)) + \int_0^t b(r; x_{r+T}(\gamma))dr$$

$$+\int_0^t \sigma(r; x_{r+T}(\gamma))d\widetilde{w}(r)\,. \tag{3.5}$$

Hence $(w(t), \{x(t;\gamma)\}_{t\geq 0})$ and $(\widetilde{w}(t), \{x(t+T;\gamma)\}_{t\geq 0})$ are two weak solutions of (2.5) on the same complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$.

By the property of Brownian motion, $(w(t)_{t\geq 0})$ and $(\widetilde{w}(t)_{t\geq 0})$ have the same distribution. Notice that the solution for NSFDEwID (3.1) is a pathwise unique strong solution, moreover NSFDEwIDs (3.1) and (3.5) have the same formation, hence there is a measurable function F such that

$$x(t;\gamma) = F(w(s); \forall s \leq t), P - a.s.$$

This means that for (3.5) we must also have

$$x(t+T;\gamma) = F(\widetilde{w}(s); \forall s \leq t), P - a.s.$$

Therefore$\forall A \in B((R^d)^n)$, and $\forall t_1, \dots, t_n \in [0,\infty)$,

$$P((x(t_1;\gamma), \dots, x(t_n;\gamma) \in A) = P((x(t_1+T;\gamma), \dots, x(t_n+T;\gamma) \in A),$$

moreover, initial values $x(T;\gamma)$ and γ have the same distribution, hence we can get that $x(T;\gamma)$ and $x(t+T;\gamma)$ have the same finite-dimensional distributions. Since γ is unique, we know that $x(t;\gamma)$ is the unique T-periodic stochastic periodic solution of (2.4) and the proof is complete.

Remark 3.6 If the solutions of the system (2.5) have the property (2.13) and (2.23), then (2.5) admits a unique stochastic periodic solution.

4 Example

In this section, to address the validity of the theory by applying the assumptions **(A1)**, **(A2)** and the Remark 3.6 we introduce an example of the system (2.5) which $w(t)$ is a one-dimensional Brownian motion.

Example 1: Consider the one-dimensional type of neutral SFDE with infinity delay:

$$d\left[x(t) - \frac{1}{2}\int_{-\infty}^{0} e^{2q\theta} D(\phi)d\theta\right] = -a|sin(t)|\phi dt + \Big[|sin(t)|\phi + \int_{-\infty}^{0} e^{2q\theta}|sin(t)|\phi(\theta)d\theta dw(t)\Big], \tag{4.1}$$

With initial value $x(t) = \xi$ when$t \in (-\infty, 0]$. Where a, q are positive numbers, $\phi \in C_r$ and $w(t)$ is a Brownian motion. It is clear that the equation (4.1) is periodic with $T = 2\pi$. Anyway, by Hölder inequality for any $\mu(d\theta) = e^{2q\theta}d\theta$ and $q > \frac{1}{8}$, it is easy to check that:

$$|D(\varphi) - D(\phi)|^2 \leq \frac{1}{8q}\int_{-\infty}^{0} e^{4q\theta}|\varphi(\theta) - \phi(\theta)|^2 d\theta.$$

Similarly, for any ϕ and $\varphi \in C_r$, define

$b(\phi) = -a|sin(t)|\phi$, $\sigma(\phi) = |sin(t)|\phi + \int_{-\infty}^{0} e^{2q\theta}|sin(t)|\phi(\theta)d\theta,$ we can show that:

$$[\varphi(0) - \phi(0) - (D(\varphi) - D(\phi))]^T[b(\varphi) - b(\phi)] = [\varphi(0) - \phi(0) - \frac{1}{2}\int_{-\infty}^{0} e^{2q\theta}(\varphi(\theta) - \phi(\theta))d\theta\,]^T[-a|sin(t)|\varphi + a|sin(t)|\phi] \leq -a|\varphi(0) - \phi(0)|^2 + \frac{a}{2}\int_{-\infty}^{0} e^{2q\theta}|\varphi(\theta) - \phi(\theta)\,|^2 d\theta, \tag{4.2}$$

And

$$|\sigma(\varphi) - \sigma(\phi)|^2 = ||sin(t)|\varphi - |sin(t)|\phi + \int_{-\infty}^{0} e^{2q\theta}(\varphi(\theta) - \phi(\theta))d\theta|^2$$

$$\leq (1+\varepsilon)||sin(t)|\varphi - |sin(t)|\phi|^2 + \frac{1+\varepsilon}{\varepsilon}|\int_{-\infty}^{0} e^{2q\theta}(\varphi(\theta) - \phi(\theta))d\theta\,|^2$$

$$\leq (1+\varepsilon)|\varphi(0)-\phi(0)|^2 + \frac{1+\varepsilon}{\varepsilon}\int_{-\infty}^{0} e^{4p\theta}|\varphi(\theta)-\phi(\theta)|^2 d\theta \tag{4.3}$$

Thus, from (4.2), (4.3) we get that: $\lambda_1 = a, \lambda_2 = \frac{a}{2}, \lambda_3 = 1+\varepsilon$ and $\lambda_4 = \frac{1+\varepsilon}{\varepsilon}$. Hence, if $\varepsilon = \frac{1}{8q}$ with $q > \frac{1}{8}$, $2\lambda_1 > 73\lambda_3 + 2\lambda_2\mu^{(2r)} + 73\lambda_4\mu^{(2r)}$ where $\mu^{(2r)} \in M_{2r}$ and

$$a > \frac{73(1+\frac{1}{8q})+73(1+8q)\mu^{(2r)}}{4-\mu^{(2r)}},$$

The $T = 2\pi$ periodic functions b, σ and the function $D(\phi)$ are satisfy the assumptions **(A1)** and **(A2)**. Now, suppose that for any $t \geq 0$,

$$b(t;x_t) = -a|sin(t)|x(t),$$
$$D(t;x_t) = -\frac{1}{2}\int_{-\infty}^{t} e^{2q(s-t)}ds$$
$$= -\frac{1}{4q}$$

And

$$\sigma(t;x_t) = |sin(t)|x(t) + \int_{-\infty}^{t} e^{2q\theta}|sin(t)|ds$$

$$= |sin(t)|x(t) + \int_{-\infty}^{t} e^{2q(s-t)}|sin(t)|ds$$

$$= |sin(t)|x(t) + \frac{|sin(t)|}{2q},$$

Define

$$V(x(t)-D(t;x_t)) = U(x(t)-D(t;x_t)) = c|x-\frac{1}{4q}|^2$$

Compute $LV(x;t)$ associated with the equation (4.1) as

$$V_t(x(t)-D(t;x_t)) = 0,$$
$$V_x(x(t)-D(t;x_t)) = 2c(x-\frac{1}{4q}),$$
$$V_{xx}(x(t)-D(t;x_t)) = 2c.$$

$$LV(x(t)-D(t;x_t)) = V_t(x(t)-D(t;x_t)) + V_x(x(t)-D(t;x_t))b(t;x_t) + \frac{1}{2}trace[\sigma^T(t;x_t)V_{xx}(x(t)-D(t;x_t))\sigma(t;x_t)] = 2c(x-\frac{1}{4q})(-a|sin(t)|x) + c(|sin(t)|x+\frac{|sin(t)|}{2q})^2 \leq -2cax^2 + \frac{ca}{2q}x + cx^2 + \frac{c}{q}x + \frac{c}{4q^2}$$
$$= c(-2a+1)x^2 + c(\frac{a+2}{2q})x + \frac{c}{4q^2} . \tag{4.4}$$

Since $(1-2a) < 0$, we have

$$\frac{1}{c(1-2a)} LV\big(x(t)-D(t;x_t)\big) \geq x^2 + \frac{(2+a)}{(1-2a)2q}x + \frac{1}{(1-2a)4q^2}$$
$$= x^2 - \frac{(2+a)}{(2a-1)2q}x + \frac{1}{(1-2a)4q^2} + (\frac{(2+a)}{(2a-1)4q})^2 - (\frac{(2+a)}{(2a-1)4q})^2$$
$$= (x-\frac{(2+a)}{(2a-1)4q})^2 + +(\frac{(2+a)}{(2a-1)4q})^2, \tag{4.5}$$

Thus, for $a = 3$

$$LV(x(t)-D(t;x_t)) \leq -\lambda(x-\frac{1}{4q})^2 + \beta \tag{4.6}$$

Where, $\lambda = c(2a-1)$ and $\beta = \frac{1}{80q^2}$. Similarly, compute $LU(x;y;t)$ associated with the equation (4.1) as

$$LU(x(t)-D(t;x_t)-y(t)+D(t;y_t)) =$$

$$2c(x-y)(-a|\sin(t)|(x-y)) + c(|\sin(t)|(x-y))^2 \leq -2ca(x-y)^2 + c(x-y)^2$$
$$= -\lambda(x-y)^2. \quad (4.7)$$

Therefore, because of **(A1)**, **(A2)** and the conditions of Lemma 2.3 and Lemma 2.5 are satisfied, an application of Theorem 3.5 yields that the system (2.3) has a unique stochastic periodic solution.

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المستخلص

الهدف من هذا العمل هو دراسة الحل الدوري -T لنوع محايد من المعادلات التفاضلية الدالية العشوائية مع تأخر لانهائي (NSFDEwID) ، حيث استخدمنا طريقة Lyapunov الثانية لإظهار محدودية الحل $x(t)$ وخريطة الحل x_t إلى المعادلات أعلاه . في هذا العمل تم استخدام مبدأ تعيين الانكماش ونظرية Banach للنقطة الصامدة. في نهاية البحث استخدمنا مثال لتوضيح نتائج العمل.