Large (k,n)-arcs in the Projective Plane of Order 37

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ABSTRACT

In this study, we use the concept of the cyclic projectivity to find some complete cyclic (k_d, n_d)-arcs that correspond to some divisors of $|\mathbb{P}^2_{37}|$. The cyclic complete arcs $(469,16)$, $(201,8)$, $(67,4)$, and $(21,2)$ are constructed. Moreover, we find 6 types of large complete $(488,17)$-arcs that containing the cyclic arc $(469,16)$, 19 types of large complete $(226,9)$-arcs that containing the cyclic arc $(201,8)$, 16 types of large complete $(91,5)$-arcs that containing the cyclic arc $(67,4)$, and 3 types of large complete $(46,3)$-arcs that containing the cyclic arc $(21,2)$.

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1. Introduction

A (k,n)-arc $\mathcal{K}$ in a projective plane $\mathbb{P}_q^2$ is a set of k points with at most n points on any line of the plane, where $n \geq 2$. Although some complete large (k,n)-arcs come from the algebraic curves of order n in $\mathbb{P}_q^2$, little results is known about (k,n)-arcs, in general, some results are known for (k,n)-arcs. In fact, the maximum value that k can take for given q and n, relatively, is known in some cases.. Throughout, $\mathbb{P}_q^2$ will denote for the projective plane $PG(2,q)$. A line ℓ of $\mathbb{P}_q^2$ is an μ-secant of a (k,n)-arc $\mathcal{K}$ if ℓ intersects $\mathcal{K}$ in μ points. Let τ_i be the total number of i-secants to $\mathcal{K}$. A (k,n)-arc is complete if there is no $(k+1,n)$-arc containing it [19-20].

Lemma 1.1. ([1]) *For a (k,n)-arc $\mathcal{K}$, the following equations hold:*

$$\sum_{i=0}^{n} \tau_i = q^2 + q + 1; \qquad 1$$

$$\sum_{i=1}^{n} i\tau_i = k(q+1); \qquad 2$$

$$\sum_{i=2}^{n} i(i-1)\tau_i = k(k-1); \qquad 3$$

C. Kestenband [2], J. C. Fisher, J. W. P. Hirschfeld, and J. A. Thas [3], E. Boros, and T. Szőnyi [4] constructed complete (q^2-q+1)-arcs in $\mathbb{P}_{q^2}^2, q \geq 3$. One of the interesting properties of theses arcs is the fact that they are fixed by a cyclic projective group of order q^2-q+1. In [5], R. H. Bruck noted that every finite Desarguesian plane of order m^2 can be partitioned into disjoint sub-planes of order m, and each subset of this partition form a complete arc of some degree. His method follows straightforward from the discovery by J. Singer that every such plane is cyclic and may be obtained from a difference set (6). In [7], P. Yff discussed the partition of the Desarguesian projective plane of order m^2, which contains $N = n_1 n_2$ points and N lines, where $N = q^2 + q + 1, n_1 = m^2 - m + 1, n_2 = m^2 + m + 1$, and m is a prim power. For the partition discussed by Yff, some members of the partition form complete arcs of large size. Other partitions have been found for $q = 3,4,5,7$ [7]. A $(k,2)$-arc is called cyclic if it consists of the points of a point orbit under a cyclic collineation group G of $\mathbb{P}_q^2$.

The complete cyclic $(k,2)$-arcs have been thoroughly investigated in [8]; the authors showed that in most cases the group G has to be a subgroup of a Singer group of $\mathbb{P}_q^2$. Such cyclic arcs will be called arcs of Singer type. Exploring arcs of large size of Singer type two cases are distinguished according as q is a square or not. In the former case, an infinite family of such arcs is well known: for each k divides $q-\sqrt{q}+1$, there exists a k-arc of Singer type [9-12]. Also, in $\mathbb{P}_q^2$, q even square, $q \geq 16$, the $q-\sqrt{q}+1$-arcs of Singer type are complete and are the largest arcs which are not contained in a $(q+2)$-arc [3]. For more information about cyclic arcs, one can see [12, 13]. The problem then to be considered is whether there exist other types of complete cyclic (k,n)-arcs of degree $n>2$; that is, complete cyclic $(k,n>2)$-arcs which induced a partition of $\mathbb{P}_q^2$. For more information about large complete arcs one can see [14-17]. The present paper begins investigating the existence of large cyclic arcs which depend on the left cosets of a normal subgroup of $\mathbb{P}_{37}^2$. Moreover, some of these cosets form complete arcs of degree bigger than 2. Also, from these cyclic (k,n)-arcs, we construct (k',n')-arcs of large size.

For a (k,n)-arc $\mathcal{K}$, if the only i are for which $\tau_i \neq 0$ are $m_1 < m_2 < \cdots < m_r < n$, then $\mathcal{K}$ is of type $(m_1, m_2, \ldots, m_r, n)$ [18].

Definition 1.1. A (k,n)-arc $\mathcal{K}$ in $\mathbb{P}_q^2$ of type $(m_1, m_2, \ldots, m_r, n)$ such that $\tau_i = t_i$, $i = m_1, m_2, \ldots, m_r, n$ is said to be has secant-distribution $({t_{m_1}}^{\tau_{m_1}}, {t_{m_2}}^{\tau_{m_2}}, \ldots, {t_{m_r}}^{\tau_{m_r}}, {t_n}^{\tau_n})$.

2. Main Results

Let $\mathbb{F}_{37}$ be the Galois field of order 37. The polynomial $\wp(x) = x^3 - 2x^2 - 2x - 2$ in the ring $\mathbb{F}_{37}[x]$ is primitive polynomial over $\mathbb{F}_{37}$, since there is no $\alpha \in \mathbb{F}_{37}$ such that $\wp(\alpha) = 0$, but it has a root in the extension field ${\mathbb{F}_{37}}^3$. The corresponding companion matrix of $\wp$ is

$$C_\wp = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix}.$$

In fact, $C_\wp$ is a cyclic projectivity which permutes the points of $\mathbb{P}_{37}^2$ in a single cycle. Let θ be the number of points in $\mathbb{P}_{37}^2$; that is, $\theta = 1407$. Define an operation $*$ on the points of $\mathbb{P}_{37}^2$ as follows:

$$P_i * P_j = P_1 {C_\wp}^{i+j-2 \bmod \theta}. \qquad 4$$

Consequently, we have

$$P_1 * P_1 = P_1 {C_{\wp}}^0 = P_1 I = P_1,$$

where I is the 3×3 identity matrix.

$$P_1 * P_2 = P_1 {C_{\wp}}^1 = P_2,$$

$$\vdots \quad \cdots \quad \vdots$$

$$P_1 * P_{1406} = P_1 {C_{\wp}}^{1405} = P_{1406},$$

$$P_1 * P_{1407} = P_1 {C_{\wp}}^{1406} = P_{1407}.$$

Under the operation defined in Equation (2.1), $\mathbb{P}_{37}^2$ becomes a group with identity P_1, where $P_1 = (1,0,0)$. Let us prove that $\mathbb{P}_{37}^2$ forms a group under the binary operation $*$:

First of all, we show inductively that $P_1 {C_{\wp}}^l = P_{l+1}$ for every integer $0 \leq l \leq \theta - 1$. Note that, $P_1 {C_{\wp}}^0 = P_1 I = P_1$ and $P_1 {C_{\wp}}^1 = P_2$, because $C_{\wp}$ is a cyclic projectivity which permutes the points of $\mathbb{P}_{37}^2$ in a single cycle. Now, suppose that $P_1 {C_{\wp}}^j = P_{j+1}$, then

$P_1 {C_{\wp}}^{j+1} = (P_1 {C_{\wp}}^j) {C_{\wp}}^1 = P_{j+1} C_{\wp} = P_{j+2}$, because $C_{\wp}$ permutes the points of $\mathbb{P}_{37}^2$ in a single cycle. Thus, $P_1 {C_{\wp}}^l = P_{l+1}$ for every integer $0 \leq l \leq \theta - 1$.

1. $\mathbb{P}_{37}^2$ is closed under $*$: if $P_i, P_j \in \mathbb{P}_{37}^2$, then $P_i * P_j = P_1 {C_{\wp}}^{i+j-2 \bmod \theta}$ according to Equation 2.1. But $0 \leq i + j - 2 \bmod \theta \leq \theta - 1$, so there is an integer $0 \leq l \leq \theta - 1$ with $l = i + j - 2 \bmod \theta$ and $P_i * P_j = P_1 {C_{\wp}}^l = P_{l+1} \in \mathbb{P}_{37}^2$ since $1 \leq l + 1 \leq \theta$.
2. $*$ is an associative binary operation on $\mathbb{P}_{37}^2$:

Assume that $P_i, P_j, P_l \in \mathbb{P}_{37}^2$. Then

$(P_i * P_j) * P_l = (P_1 {C_{\wp}}^{i+j-2 \bmod \theta}) * P_l = P_{i+j-1 \bmod \theta} * P_l = P_1 {C_{\wp}}^{i+j+l-3 \bmod \theta}$. On the other hand, we have $P_i * (P_j * P_l) = P_i * (P_1 {C_{\wp}}^{j+l-2 \bmod \theta}) = P_i * P_{j+l-1 \bmod \theta} = P_1 {C_{\wp}}^{i+j+l-3 \bmod \theta}$. So, $(P_i * P_j) * P_l = P_i * (P_j * P_l)$ for all $P_i, P_j, P_l \in \mathbb{P}_{37}^2$.

3. Existences of the identity: The point $P_1 = (1,0,0)$ acts as the identity element for the operation $*$. Note that

$P_1 * P_j = P_j * P_1 = P_1 {C_{\wp}}^{j+1-2 \bmod \theta} = P_1 {C_{\wp}}^{j-1 \bmod \theta} = P_j$, because $C_{\wp}$ permutes the points of $\mathbb{P}_{37}^2$ in a single cycle.

4. Existences of the inverses: let $P_i \in \mathbb{P}_{37}^2$, where $1 \leq i \leq \theta$. Then $P_{2-i \bmod \theta} \in \mathbb{P}_{37}^2$ and

$P_i * P_{2-i \bmod \theta} = P_{2-i \bmod \theta} * P_i = P_1 {C_{\wp}}^0 = P_i I = P_1.$
In fact, $\mathbb{P}^2_{37}$ is a cyclic group generated by P_2. More precisely,

$\mathbb{P}^2_{37} = \langle P_m \rangle$, for all positive integer ν such that $2 \leq m \leq 1407$ and $\gcd(m, 1407) = 1$.
Note that, P_2 in $\mathbb{P}^2_{37}$ and

$$\begin{aligned}
P_2^0 &= identity = P_1,\\
P_2^1 &= P_2,\\
P_2^2 &= P_2 * P_2 = P_1 {C_{\wp}}^{2 \bmod \theta} = P_3,\\
P_2^3 &= P_2^2 * P_2 = P_3 * P_2 = P_1 {C_{\wp}}^{3 \bmod \theta} = P_4,\\
&\vdots \quad \vdots \quad \vdots\\
P_2^{\alpha-1} &= P_2^{\alpha-2} * P_2 = P_{\alpha-1} * P_2 = P_1 {C_{\wp}}^{\alpha-1 \bmod \theta} = P_\alpha.
\end{aligned}$$

So, we prove inductively that $P_\alpha = P_2^{\alpha-1}$ for every $\alpha = 1,2,3,\dots,1407$.
Consequently, we have

$$P_\alpha{}^{\nu} * P_\beta{}^{\mu} = ({P_2}^{\alpha-1})^{\nu} * ({P_2}^{\beta-1})^{\mu} = P_2^{(\nu\alpha+\mu\beta)-(\nu+\mu) \bmod \theta}, \qquad 5$$

for each P_α, P_β in $\mathbb{P}^2_{37}$ and $n, m \in \mathbb{Z}$. The inverse of P_α with respect to the operation $*$ is P_β, where $\alpha + \beta = 2 \bmod \theta$.

2.1 Large arcs from cyclic arc of size 469 and degree 16

According to Lagrange theorem, there exist a cyclic subgroup of $\mathbb{P}^2_{37}$ of order 469, say $\mathcal{C}_{469} = \langle P_4 \rangle$. The elements $\mathcal{C}_{469}$ are the points P_i, where i takes all of the following indices:

1	4	7	10	13	16	19	22	25	28	31	34	37	40	43
46	49	52	55	58	61	64	67	70	73	76	79	82	85	88
91	94	97	100	103	106	109	112	115	118	121	124	127	130	
133	136	139	142	145	148	151	154	157	160	163	166	169	172	
175	178	181	184	187	190	193	196	199	202	205	208	211	214	
217	220	223	226	229	232	235	238	241	244	247	250	253	256	
259	262	265	268	271	274	277	280	283	286	289	292	295	298	
301	304	307	310	313	316	319	322	325	328	331	334	337	340	
343	346	349	352	355	358	361	364	367	370	373	376	379	382	
385	388	391	394	397	400	403	406	409	412	415	418	421	424	
427	430	433	436	439	442	445	448	451	454	457	460	463	466	
469	472	475	478	481	484	487	490	493	496	499	502	505	508	
511	514	517	520	523	526	529	532	535	538	541	544	547	550	
553	556	559	562	565	568	571	574	577	580	583	586	589	592	

595	598	601	604	607	610	613	616	619	622	625	628	631	634
637	640	643	646	649	652	655	658	661	664	667	670	673	676
679	682	685	688	691	694	697	700	703	706	709	712	715	718
721	724	727	730	733	736	739	742	745	748	751	754	757	760
763	766	769	772	775	778	781	784	787	790	793	796	799	802
805	808	811	814	817	820	823	826	829	832	835	838	841	844
847	850	853	856	859	862	865	868	871	874	877	880	883	886
889	892	895	898	901	904	907	910	913	916	919	922	925	928
931	934	937	940	943	946	949	952	955	958	961	964	967	970
973	976	979	982	985	988	991	994	997	1000	1003	1006	1009	
1012	1015	1018	1021	1024	1027	1030	1033	1036	1039	1042	1045		
1048	1051	1054	1057	1060	1063	1066	1069	1072	1075	1078	1081		
1084	1087	1090	1093	1096	1099	1102	1105	1108	1111	1114	1117		
1120	1123	1126	1129	1132	1135	1138	1141	1144	1147	1150	1153		
1156	1159	1162	1165	1168	1171	1174	1177	1180	1183	1186	1189		
1192	1195	1198	1201	1204	1207	1210	1213	1216	1219	1222	1225		
1228	1231	1234	1237	1240	1243	1246	1249	1252	1255	1258	1261		
1264	1267	1270	1273	1276	1279	1282	1285	1288	1291	1294	1297		
1300	1303	1306	1309	1312	1315	1318	1321	1324	1327	1330	1333		
1336	1339	1342	1345	1348	1351	1354	1357	1360	1363	1366	1369		
1372	1375	1378	1381	1384	1387	1390	1393	1396	1399	1402	1405		

The points of $\mathcal{C}_{469}$ forms a complete cyclic (469,16)-arc in $\mathbb{P}_{37}^2$ with secant-distribution $(469^{\tau_9}, 469^{\tau_{13}}, 469^{\tau_{16}})$.

Theorem 2.1 *In $\mathbb{P}_{37}^2$, there exist 6 types of a complete (488,17)-arc, containing $\mathcal{C}_{469}$, up to secant-distributions.*

Proof. Consider the following 19 points of $\mathbb{P}_{37}^2 \backslash \mathcal{C}_{469}$, say

18	26	47	248	258	326	537	561	618	891	921	1161	1185	1218	1241	1259
1364	1394	1406													

By adding these points to the cyclic arc $\mathcal{C}_{469}$, we get a complete (488,17)-arc with secant-distribution $(267^{\tau_9}, 146^{\tau_{10}}, 43^{\tau_{11}}, 11^{\tau_{12}}, 292^{\tau_{13}}, 131^{\tau_{14}}, 34^{\tau_{15}}, 275^{\tau_{16}}, 208^{\tau_{17}})$. In fact, $\mathcal{C}_{469}$ is complete because we cannot added any other point to it. Table 1 illustrates all types of the complete (488,17)-arcs, that constructed from (469,16)-arc $\mathcal{C}_{469}$, with their secant-distributions.

Table 1. Complete $(488, 17)$-arcs in $\mathbb{P}^2_{37}$

$\mathcal{A}_{469} = \{$Point of $\mathbb{P}^2_{37} \backslash \mathcal{C}_{469}\}$	Description for $\mathcal{C}_{469} \cup \mathcal{A}_{469}$
18 26 47 248 258 326 537 561 618 891 921 1161 1185 1218 1241 1259 1364 1394 1406	Complete (488,17)-arc with secant-distribution $(267^{\tau_9}, 146^{\tau_{10}}, 43^{\tau_{11}}, 11^{\tau_{12}}, 292^{\tau_{13}}, 131^{\tau_{14}}, 34^{\tau_{15}}, 275^{\tau_{16}}, 208^{\tau_{17}})$
18 26 47 248 258 326 561 618 876 891 921 1161 1185 1218 1241 1259 1364 1394 1406	Complete (488,17)-arc with secant-distribution $(271^{\tau_9}, 141^{\tau_{10}}, 42^{\tau_{11}}, 12^{\tau_{12}}, 290^{\tau_{13}}, 133^{\tau_{14}}, 38^{\tau_{15}}, 273^{\tau_{16}}, 207^{\tau_{17}})$
18 26 47 248 258 326 561 618 876 891 921 1185 1218 1241 1259 1364 1383 1394 1406	Complete (488,17)-arc with secant-distribution $(271^{\tau_9}, 139^{\tau_{10}}, 44^{\tau_{11}}, 14^{\tau_{12}}, 290^{\tau_{13}}, 131^{\tau_{14}}, 36^{\tau_{15}}, 275^{\tau_{16}}, 207^{\tau_{17}})$
18 26 47 248 258 326 447 537 561 618 891 921 1161 1185 1241 1259 1364 1394 1406	Complete (488,17)-arc with secant-distribution $(269^{\tau_9}, 139^{\tau_{10}}, 50^{\tau_{11}}, 10^{\tau_{12}}, 293^{\tau_{13}}, 127^{\tau_{14}}, 36^{\tau_{15}}, 275^{\tau_{16}}, 208^{\tau_{17}})$
18 26 47 248 258 326 447 561 618 876 891 921 1161 1185 1241 1259 1364 1394 1406	Complete (488,17)-arc with secant-distribution $(272^{\tau_9}, 136^{\tau_{10}}, 48^{\tau_{11}}, 11^{\tau_{12}}, 291^{\tau_{13}}, 130^{\tau_{14}}, 38^{\tau_{15}}, 274^{\tau_{16}}, 207^{\tau_{17}})$

18 26 47 248 258 326 447 561 618 876 891 921 1185 1241 1259 1364 1383 1394 1406	Complete (488,17)-arc with secant-distribution $(271^{\tau_9}, 137^{\tau_{10}}, 48^{\tau_{11}}, 12^{\tau_{12}}, 292^{\tau_{13}}, 127^{\tau_{14}}, 38^{\tau_{15}}, 275^{\tau_{16}}, 207^{\tau_{17}})$

2.2 Large arcs from cyclic arc of size 201 and degree 8

Lagrange theorem tell us that there exists a cyclic subgroup of $\mathbb{P}^2_{37}$ of order 201, say $\mathcal{C}_{201} = \langle P_8 \rangle$. Let us list some elements in this subgroup using Equation (2.2):

$$\begin{aligned}
{P_8}^2 &= P_8 * P_8 = P_2^{14} = P_{15},\\
{P_8}^3 &= P_8 * P_{15} = P_2^{21} = P_{22},\\
&\vdots \quad \cdots \quad \vdots\\
{P_8}^{200} &= (P_2^7)^{200} = {P_2}^{1400} = P_{1401},\\
{P_8}^{201} &= P_2^{1407} = P_1.
\end{aligned}$$

The elements $\mathcal{C}_{201}$ are the points P_i, where i takes all of the following indices:

1	8	15	22	29	36	43	50	57	64	71	78	85	92	99
106	113	120	127	134	141	148	155	162	169	176	183	190	197	
204	211	218	225	232	239	246	253	260	267	274	281	288	295	
302	309	316	323	330	337	344	351	358	365	372	379	386	393	
400	407	414	421	428	435	442	449	456	463	470	477	484	491	
498	505	512	519	526	533	540	547	554	561	568	575	582	589	
596	603	610	617	624	631	638	645	652	659	666	673	680	687	
694	701	708	715	722	729	736	743	750	757	764	771	778	785	
792	799	806	813	820	827	834	841	848	855	862	869	876	883	
890	897	904	911	918	925	932	939	946	953	960	967	974	981	
988	995	1002	1009	1016	1023	1030	1037	1044	1051	1058	1065	1072		
1079	1086	1093	1100	1107	1114	1121	1128	1135	1142	1149	1156			
1163	1170	1177	1184	1191	1198	1205	1212	1219	1226	1233	1240			
1247	1254	1261	1268	1275	1282	1289	1296	1303	1310	1317	1324			
1331	1338	1345	1352	1359	1366	1373	1380	1387	1394	1401				

The points of $\mathcal{C}_{201}$ forms a complete cyclic (201,8)-arc in $\mathbb{P}_{37}^2$ with secant-distribution $(603^{\tau_3}, 603^{\tau_7}, 201^{\tau_8})$.

Theorem 2.2 *In $\mathbb{P}_{37}^2$, there exist 19 types of a complete (226,9)-arc, containing $\mathcal{C}_{201}$, up to secant-distributions.*

Proof. Consider the following 25 points of $\mathbb{P}_{37}^2 \backslash \mathcal{C}_{201}$, say

23	61	124	163	191	195	220	226	229	254	577	684	772	850	1015	1020
1090	1104	1123	1209	1224	1276	1360	1396	1405							

By adding these points to the cyclic arc $\mathcal{C}_{201}$, we get a complete (226,9)-arc with secant-distribution $(312^{\tau_3}, 186^{\tau_4}, 81^{\tau_5}, 20^{\tau_6}, 287^{\tau_7}, 315^{\tau_8}, 206^{\tau_9})$. By adding points of $\mathbb{P}_{37}^2 \backslash \mathcal{C}_{201}$ to the complete cyclic (201,8)-arc $\mathcal{C}_{201}$, we have all types of the complete (226,9)-arcs with their secant-distributions as shown in Table 2.

Table 2. Complete $(226, 9)$-arcs in $\mathbb{P}_{37}^2$

$\boldsymbol{\mathcal{A}_{201}}$ = {Point of $\mathbb{P}_{37}^2 \backslash \mathcal{C}_{201}$}	Description for $\mathcal{C}_{201} \cup \boldsymbol{\mathcal{A}_{201}}$
23 61 124 163 191 195 220 226 229 254 577 684 772 850 1015 1020 1090 1104 1123 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(312^{\tau_3}, 186^{\tau_4}, 81^{\tau_5}, 20^{\tau_6}, 287^{\tau_7}, 315^{\tau_8}, 206^{\tau_9})$
23 61 124 163 191 195 220 226 229 254 577 684 772 850 1015 1020 1060 1090 1104 1209 1224 1276 1314 1396 1405	Complete (226,9)-arc with secant-distribution $(311^{\tau_3}, 185^{\tau_4}, 85^{\tau_5}, 19^{\tau_6}, 291^{\tau_7}, 305^{\tau_8}, 211^{\tau_9})$
23 61 124 163 191 195 220 226 229 254 577 684 772 850 1015 1020	Complete (226,9)-arc with secant-distribution $(309^{\tau_3}, 190^{\tau_4}, 82^{\tau_5}, 18^{\tau_6}, 292^{\tau_7}, 305^{\tau_8}, 211^{\tau_9})$

1090 1104 1209 1224 1276 1314 1360 1396 1405	
23 61 124 163 191 195 220 226 254 577 684 772 850 950 1015 1020 1090 1104 1209 1224 1276 1314 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(310^{\tau_3}, 189^{\tau_4}, 82^{\tau_5}, 17^{\tau_6}, 290^{\tau_7}, 311^{\tau_8}, 208^{\tau_9})$
61 124 163 191 195 220 226 229 254 577 684 772 850 950 1015 1020 1090 1104 1123 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(313^{\tau_3}, 185^{\tau_4}, 80^{\tau_5}, 21^{\tau_6}, 285^{\tau_7}, 319^{\tau_8}, 204^{\tau_9})$
61 124 163 191 195 220 226 229 254 577 684 772 850 950 1015 1020 1090 1104 1209 1224 1276 1314 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(310^{\tau_3}, 190^{\tau_4}, 80^{\tau_5}, 18^{\tau_6}, 289^{\tau_7}, 313^{\tau_8}, 207^{\tau_9})$
23 124 163 191 195 220 226 254 327 577 684 772 850 950 1015 1020 1090 1104 1123 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(311^{\tau_3}, 188^{\tau_4}, 82^{\tau_5}, 16^{\tau_6}, 288^{\tau_7}, 317^{\tau_8}, 205^{\tau_9})$
23 124 163 191 195 220 226 254 327 577 684 772 850 950 1015 1020	Complete (226,9)-arc with secant-distribution $(307^{\tau_3}, 195^{\tau_4}, 81^{\tau_5}, 13^{\tau_6}, 293^{\tau_7}, 309^{\tau_8}, 209^{\tau_9})$

1090 1104 1209 1224 1276 1314 1360 1396 1405	
23 124 191 195 220 226 254 327 395 577 684 772 850 950 1015 1020 1090 1104 1123 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(308^{\tau_3}, 196^{\tau_4}, 75^{\tau_5}, 18^{\tau_6}, 289^{\tau_7}, 315^{\tau_8}, 206^{\tau_9})$
61 124 163 191 195 220 226 254 577 684 772 850 950 1015 1020 1034 1090 1104 1144 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(311^{\tau_3}, 193^{\tau_4}, 71^{\tau_5}, 23^{\tau_6}, 283^{\tau_7}, 325^{\tau_8}, 201^{\tau_9})$
61 124 163 191 195 220 226 254 577 684 772 850 950 1015 1020 1090 1104 1123 1144 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(312^{\tau_3}, 190^{\tau_4}, 74^{\tau_5}, 22^{\tau_6}, 283^{\tau_7}, 325^{\tau_8}, 201^{\tau_9})$
124 163 191 195 220 226 254 327 577 684 772 850 950 1015 1020 1090 1104 1123 1144 1209 1224 1276 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(311^{\tau_3}, 190^{\tau_4}, 79^{\tau_5}, 16^{\tau_6}, 286^{\tau_7}, 323^{\tau_8}, 202^{\tau_9})$
124 191 195 220 226 254 327 395 577 684 772 850 950 1015	Complete (226,9)-arc with secant-distribution $(308^{\tau_3}, 196^{\tau_4}, 76^{\tau_5}, 16^{\tau_6}, 289^{\tau_7}, 317^{\tau_8}, 205^{\tau_9})$

1020 1090 1104 1123 1144 1209 1224 1276 1360 1396 1405	
61 124 163 191 195 220 226 254 577 684 772 850 950 1015 1020 1090 1104 1144 1209 1224 1276 1314 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(309^{\tau_3}, 195^{\tau_4}, 74^{\tau_5}, 20^{\tau_6}, 284^{\tau_7}, 322^{\tau_8}, 203^{\tau_9})$
124 163 191 195 220 226 254 327 577 684 772 850 950 1015 1020 1090 1104 1144 1209 1224 1276 1314 1360 1396 1405	Complete (226,9)-arc with secant-distribution $(308^{\tau_3}, 195^{\tau_4}, 79^{\tau_5}, 14^{\tau_6}, 287^{\tau_7}, 320^{\tau_8}, 204^{\tau_9})$
23 61 124 163 191 195 220 226 254 577 684 772 850 950 1015 1020 1060 1090 1104 1209 1224 1276 1314 1396 1405	Complete (226,9)-arc with secant-distribution $(312^{\tau_3}, 183^{\tau_4}, 87^{\tau_5}, 17^{\tau_6}, 290^{\tau_7}, 309^{\tau_8}, 209^{\tau_9})$
61 124 163 191 195 220 226 229 254 577 684 772 850 950 1015 1020 1060 1090 1104 1209 1224 1276 1314 1396 1405	Complete (226,9)-arc with secant-distribution $(313^{\tau_3}, 183^{\tau_4}, 84^{\tau_5}, 19^{\tau_6}, 287^{\tau_7}, 315^{\tau_8}, 206^{\tau_9})$
23 124 163 191 195 220 226 254 327 577 684 772 850 950 1015 1020	Complete (226,9)-arc with secant-distribution $(309^{\tau_3}, 190^{\tau_4}, 84^{\tau_5}, 14^{\tau_6}, 292^{\tau_7}, 309^{\tau_8}, 209^{\tau_9})$

1060 1090 1104 1209 1224 1276 1314 1396 1405	
23 124 191 195 220 226 254 327 577 684 772 850 950 1015 1020 1060 1090 1104 1124 1209 1224 1276 1314 1396 1405	Complete (226,9)-arc with secant-distribution $(303^{\tau_3}, 203^{\tau_4}, 77^{\tau_5}, 13^{\tau_6}, 297^{\tau_7}, 301^{\tau_8}, 213^{\tau_9})$

2.3 Large arcs from cyclic arc of size 67 and degree 4

Being 67 is a factor of $|\mathbb{P}^2_{37}|$, there is there is a cyclic subgroup of $\mathbb{P}^2_{37}$ of order 67, say $\mathcal{C}_{67} = \langle P_{22} \rangle$. Let us list some element in this subgroup using Equation 5:

$$\begin{aligned}
{P_{22}}^2 &= P_{22} * P_{22} = P_2^{42} = P_{43}, \\
{P_{22}}^3 &= P_{22} * P_{43} = P_2^{63} = P_{64}, \\
&\vdots \quad \cdots \quad \vdots \\
{P_{22}}^{66} &= (P_2^{21})^{66} = P_2^{1386} = P_{1387}, \\
{P_{22}}^{67} &= P_2^{1407} = P_1.
\end{aligned}$$

The normal subgroup $\mathcal{C}_{67}$ of $\mathbb{P}^2_{37}$ forms a complete cyclic (67,4)-arc in $\mathbb{P}^2_{37}$ with secant-distribution

$$(201^{\tau_0}, 469^{\tau_1}, 402^{\tau_2}, 67^{\tau_3}, 268^{\tau_4})$$

268 67 402 469 201

The elements $\mathcal{C}_{67}$ are the points P_i, where i takes all of the following indices:

1	22	43	64	85	106	127	148	169	190	211	232	253	274
295	316	337	358	379	400	421	442	463	484	505	526	547	568
589	610	631	652	673	694	715	736	757	778	799	820	841	862
883	904	925	946	967	988	1009	1030	1051	1072	1093	1114	1135	
1156	1177	1198	1219	1240	1261	1282	1303	1324	1345	1366	1387		

Theorem 2.3 *In $\mathbb{P}^2_{37}$, there exist 16 types of a complete (91,5)-arc, containing $\mathcal{C}_{67}$, up to secant-distributions.*

Proof. Consider the following 24 points of $\mathbb{P}^2_{37} \backslash \mathcal{C}_{67}$, say

33	42	50	80	87	90	147	206	310	312	338	356	386	495	554	583	651
875	929	1065	1142	1150	1268	13944										

By adding these points to the cyclic arc $\mathcal{C}_{67}$, we get a complete (91,5)-arc with secant-distribution $(114^{\tau_0}, 293^{\tau_1}, 411^{\tau_2}, 202^{\tau_3}, 198^{\tau_4}, 189^{\tau_5})$. All types of the complete (91,5)-arcs with their secant-distributions are shown in Table 3.

Table 3. Complete $(\mathbf{91}, \mathbf{5})$-arcs in $\mathbb{P}^2_{37}$

$\mathcal{A}_{67}$ = {Point of $\mathbb{P}^2_{37} \backslash \mathcal{C}_{67}$}	Description for $\mathcal{C}_{67} \cup \mathcal{A}_{67}$
33 42 50 80 87 90 147 206 310 312 338 356 386 495 554 583 651 875 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(116^{\tau_0}, 290^{\tau_1}, 407^{\tau_2}, 212^{\tau_3}, 192^{\tau_4}, 190^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 386 554 575 583 638 651 875 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(117^{\tau_0}, 290^{\tau_1}, 405^{\tau_2}, 208^{\tau_3}, 201^{\tau_4}, 186^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 386 495 554 575 583 651 875 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(117^{\tau_0}, 288^{\tau_1}, 408^{\tau_2}, 211^{\tau_3}, 194^{\tau_4}, 189^{\tau_5})$
33 42 50 80 87 90 134 147 206 310 338 356 386 495 554 583 651 875 902 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(115^{\tau_0}, 290^{\tau_1}, 413^{\tau_2}, 204^{\tau_3}, 195^{\tau_4}, 190^{\tau_5})$
33 42 50 80 87 90 134 147 188 206	Complete (91,5)-arcs with secant-distribution

310 386 495 554 575 583 651 875 902 1065 1142 1150 1268 1394	$(115^{\tau_0}, 287^{\tau_1}, 420^{\tau_2}, 201^{\tau_3}, 192^{\tau_4}, 192^{\tau_5})$
33 42 50 80 87 90 134 147 206 310 338 386 495 554 575 583 651 875 902 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(116^{\tau_0}, 288^{\tau_1}, 414^{\tau_2}, 203^{\tau_3}, 197^{\tau_4}, 189^{\tau_5})$
33 42 50 80 87 90 147 206 310 338 356 386 495 554 583 651 875 902 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(118^{\tau_0}, 286^{\tau_1}, 409^{\tau_2}, 210^{\tau_3}, 196^{\tau_4}, 188^{\tau_5})$
33 42 50 80 87 90 147 206 310 338 386 495 554 575 583 651 875 902 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(119^{\tau_0}, 285^{\tau_1}, 407^{\tau_2}, 212^{\tau_3}, 197^{\tau_4}, 187^{\tau_5})$
33 42 50 80 87 90 134 147 188 206 310 386 554 575 583 638 651 875 902 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(114^{\tau_0}, 293^{\tau_1}, 411^{\tau_2}, 202^{\tau_3}, 198^{\tau_4}, 189^{\tau_5})$
33 42 50 80 87 90 134 147 206 310 338 386 554 575 583 638 651 875 902 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(116^{\tau_0}, 291^{\tau_1}, 408^{\tau_2}, 203^{\tau_3}, 203^{\tau_4}, 186^{\tau_5})$
33 42 50 80 87 90 147 206 310 338 386 554 575 583 638 651 875 902 929 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(119^{\tau_0}, 288^{\tau_1}, 401^{\tau_2}, 212^{\tau_3}, 203^{\tau_4}, 184^{\tau_5})$

33 42 50 80 87 90 147 206 310 312 338 356 386 554 583 651 929 1034 1065 1133 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(116^{\tau_0}, 292^{\tau_1}, 403^{\tau_2}, 212^{\tau_3}, 196^{\tau_4}, 188^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 356 386 495 554 583 651 929 1034 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(117^{\tau_0}, 288^{\tau_1}, 405^{\tau_2}, 220^{\tau_3}, 185^{\tau_4}, 192^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 386 495 554 575 583 651 929 1034 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(117^{\tau_0}, 288^{\tau_1}, 406^{\tau_2}, 217^{\tau_3}, 188^{\tau_4}, 191^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 386 554 575 583 638 651 929 1034 1065 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(117^{\tau_0}, 290^{\tau_1}, 402^{\tau_2}, 217^{\tau_3}, 192^{\tau_4}, 189^{\tau_5})$
33 42 50 80 87 90 147 206 310 312 338 386 554 583 638 651 929 1034 1065 1133 1142 1150 1268 1394	Complete (91,5)-arcs with secant-distribution $(115^{\tau_0}, 292^{\tau_1}, 408^{\tau_2}, 207^{\tau_3}, 196^{\tau_4}, 189^{\tau_5})$

2.4 Large arcs from cyclic arc of size 21 and degree 2

There is there is a cyclic subgroup of $\mathbb{P}_{37}^2$ of order 21, since 2 is a factor of $|\mathbb{P}_{37}^2|$, , say $\mathcal{C}_{21} = \langle P_{68} \rangle$. The following are some points in $\mathcal{C}_{21}$. Using Equation 5, we have:

$$\begin{aligned}
{P_{68}}^2 &= P_{68} * P_{68} = P_2^{134} = P_{135}, \\
{P_{68}}^3 &= P_{68} * P_{134} = P_2^{201} = P_{202}, \\
&\vdots \quad \cdots \quad \vdots \\
{P_{68}}^{20} &= (P_2^{67})^{20} = P_2^{1340} = P_{1341}, \\
{P_{68}}^{21} &= P_2^{1407} = P_1.
\end{aligned}$$

The elements of $\mathcal{C}_{21}$ forms a complete cyclic (21,2)-arc with secant-distribution $(819^{\tau_0}, 378^{\tau_1}, 210^{\tau_2})$.

In fact, the elements of $\mathcal{C}_{21}$ are the points P_i, where i takes all of the following indices:

1 68 135 202 269 336 403 470 537 604 671 738 805 872 939 1006 1073 1140 1207 1274 1341

Theorem 2.4 *In $\mathbb{P}^2_{37}$, there exist 3 types of a complete (46,3)-arc, containing $\mathcal{C}_{21}$, up to secant-distributions.*

Proof. Consider the following 25 points of $\mathbb{P}^2_{37} \backslash \mathcal{C}_{21}$, say

19 52 82 319 563 580 585 697 729 753 796 844 848 969 1163 1183 1191 1193 1225 1277 1317 1346 1394 1406 1407

By adding these points to the cyclic arc $\mathcal{C}_{21}$, we get a complete (46,3)-arc with secant-distribution $(479^{\tau_0}, 323^{\tau_1}, 390^{\tau_2}, 215^{\tau_3})$. All types of the complete (46,3)-arcs with their secant-distributions are shown in Table 4.

220 375 338 474

Table 4. Complete $(46, 3)$-arcs in $\mathbb{P}^2_{37}$

$\boldsymbol{\mathcal{A}_{21}}$ = {**Point of** $\mathbb{P}^2_{37} \backslash \mathcal{C}_{21}$}	**Description for** $\mathcal{C}_{21} \cup \boldsymbol{\mathcal{A}_{21}}$
19 52 82 319 563 580 585 697 729 753 796 844 848 969 1163 1183 1191 1193 1225 1277 1317 1346 1394 1406 1407	Complete (46,3)-arcs with secant-distribution $(479^{\tau_0}, 323^{\tau_1}, 390^{\tau_2}, 215^{\tau_3})$
19 52 82 319 580 585 697 753 796 844 848 962 969 1093 1163 1183 1191 1193 1225 1277 1317 1346 1394 1406 1407	Complete (46,3)-arcs with secant-distribution $(474^{\tau_0}, 338^{\tau_1}, 375^{\tau_2}, 220^{\tau_3})$

19 52 82 319 580 585 697 796 844 848 962 969 1093 1163 1183 1191 1193 1225 1277 1317 1346 1382 1394 1406 1407	Complete (46,3)-arcs with secant-distribution $(472^{\tau_0}, 344^{\tau_1}, 369^{\tau_2}, 222^{\tau_3})$

Conclusions

In this paper, we define an operation $*$ on the points of the projective plane of oder 37. Moreover, we proved that $\mathbb{P}_{37}^2$ with this binary operation forms a group. Due to the highly parallelizable nature of our search, it is likely possible to make use of the existing search procedures to find different types of (k, n)-arcs in $\mathbb{P}_{37}^2$. In fact, we constructed a complete cyclic arcs as subgroups of our defining group. According to this method the complete cyclic arcs, say, (469,16), (201,8), (67,4), and (21,2) are constructed. Moreover, we find 6 types of large complete (488,17)-arcs from cyclic arc (469,16), 19 types of large complete (226,9)-arcs from cyclic arc (201,8), 16 types of large complete (91,5)-arcs from cyclic arc (67,4), and 3 types of large complete (46,3)-arcs from cyclic arc (21,2). The work presented in this paper opens many doors for further study. Several of these possibilities are addressed as follows. One can find large complete arcs for $q > 37$. On other hand, find different types of large complete arcs.

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اقواس عظمى من النمط (k, n) في مستو الاسقاط من الرتبة 37

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المستخلص

في هذا البحث ، استخدمنا مفهوم الاسقاط الدوار لإيجاد بعض الاقواس الدوارة من النمط (k_d, n_d) والتي تقابل بعض القواسم ل $|\mathbb{P}_{37}^{2}|$. تم انشاء الاقواس الدوارة الكاملة من النمط (469,16)، (201,8)، (67,4) و (21,2). علاوة على ذلك، وجدنا 6 انواع من الاقواس الكاملة ذات الحجم الكبير ومن النمط (488,17) والتي تحتوي على القوس الدوار (469,16)، 19 نوعا من الاقواس الكاملة ذات الحجم الكبير ومن النمط (226,9) والتي تحتوي على القوس الدوار (201,8)، 16 نوعا من الاقواس الكاملة ذات الحجم الكبير ومن النمط (91,5) والتي تحتوي على القوس الدوار (67,4)، و ثلاث انواع من الاقواس الكاملة ذات الحجم الكبير ومن النمط (46,3) والتي تحتوي على القوس الدوار (21,2).