r-Domination Number for Some Special Graphs

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ABSTRACT

In this study, bi- and triple effect- domination expands into r-domination. Given a finite, nontrivial, simple, undirected graph G with no isolated vertex, a subset $D \subseteq V$ is r-dominant if every $u \in D$ dominates r vertices from $V\backslash D$ with $r \geq 1$. $\gamma_r(G)$ represents the minimum ultimate dominant set. For specific graphs, dominance is determined.

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1. Introduction

Let G be a graph is a pair (V, E), where $V = V(G)$ is the set of vertices or points and $E = E(G)$ is the set of edges or lines and let $n = |V(G)|$ be the order of the graph G and $m = |E(G)|$ be the size of the graph G. The degree of a vertex u is the number of edges which incident on it denoted by $deg(u)$. A vertex of degree zero is an isolated vertex and a vertex of degree one is a pendant, also said end vertex or leaf. The minimum degree of a graph G denoted by $\delta(G)$ is the degree of the vertex with the least number of edges incident to it and the maximum degree of a graph G denoted by $\Delta(G)$ is the degree of the vertex with the greatest number of edges incident to it, respectively. The open neighborhood of a vertex w is the set $N(w) = \{u \in V, uw \in E(G)\}$ and closed neighborhood is the set $N[u] = N(u) \cup \{u\}$. Graph theory has several topics for more information about it see [1-3]. The study of domination problem is grown fast in graph theory. In our life, we can be representing any as a graph by represent its subjects as vertices and the communication between them represented as edges. For more information about of domination such as in [4,5]. A set $D \subseteq V$ is called a dominating set of G if every vertex in $v \in V \backslash D$ has a neighbor $u \in D$, that is $N(v) \cap D \neq \emptyset \, \forall \, v \in V \backslash D$. The domination number of a graph G is the cardinality of a minimum dominating set in G, denoted by $\gamma(G)$ and this a notation was introduced by *Cockayne* and *Hedetniemi* in 1977 [6]. A subset $D \subset V(G)$ is a bi-dominating set in G if every vertex $v \in D$ dominates exactly two vertices in $V \backslash D$, such that $|N(v) \cap V \backslash D| = 2$, the cardinality of the minimum bi-dominating set in G is known as bi-domination number of G and denoted by $\gamma_{bi}(G)$. A subset $D \subset V(G)$ is a triple effect dominating set in G if every vertex $v \in D$ dominates exactly three vertices in $V \backslash D$, such that $|N(v) \cap V \backslash D| = 3$, the cardinality of the minimum triple effect dominating set in G is known as triple effect domination number of G and denoted by $\gamma_{te}(G)$. There are different types of domination, one can see [7-9]. We introduce new type of domination in graphs in this paper called the r- domination. Each vertex in an r-dominating set dominates exactly r vertices of the remaining vertices. Some bounds on r-domination number associated with complete graph, complete bipartite graph, wheel graph, tadpole graph, lollipop graph, barbell graph, complement of path graph, cycle graph, complete bipartite graph are introduced.

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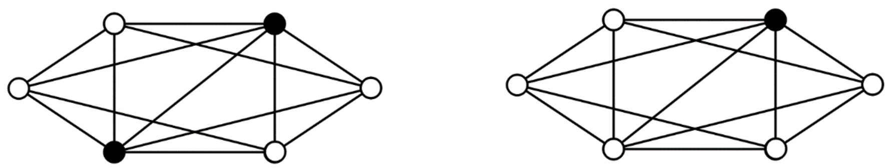
Remark 1.1

a) [9] The path graph P_n and cycle graph $C_n, n \geq 3$ has $\gamma_{bi}(P_n) = \left\lceil \frac{n}{3} \right\rceil, n \neq 4$ and $\gamma_{bi}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

b) [7,10] For a wheel graph $W_n(n \geq 3$ and $n \neq 5), \gamma_{bi}(W_n) = 2\left\lceil \frac{n}{4} \right\rceil$ and $\gamma_{te}(W_n) = \left\lceil \frac{n}{3} \right\rceil$.

Definition 1.2 Let G be a finite, nontrivial, simple and undirected graph without isolated vertex. A dominating subset $D \subseteq V$ is an r-dominating set in G if every $u \in D$ dominates r vertices from $V \backslash D$ such that $|N(u) \cap V \backslash D| = r$ where r is positive integers such that $r \geq 1$. For example, see Figure 1.

Definition 1.3 The cardinality of the minimum r-dominating set in G is known as r-domination number of G and denoted by $\gamma_r(G)$.



a) When r = 4 b) When r = 5

Figure 1: A minimum r-dominating set.

Observation 1.4 For any finite simple $G = (n, m)$ with r-dominating set D and r-domination number $\gamma_r(G)$, we have:

a) The order of G is $n \geq r + 1$.
b) $\delta(G) \geq 1$ and $\Delta(G) \geq r$.
c) Every $v \in D, deg(v) \geq r$.
d) Every support vertex $v, v \in D$.
e) $\gamma(G) \leq \gamma_r(G)$.

2. r-Domination in Graphs

Proposition 2.1 The path P_n and cycle graph C_n doesn't have an r-dominating set if $r \geq 3$.
Proof. According to Observation 1.4.

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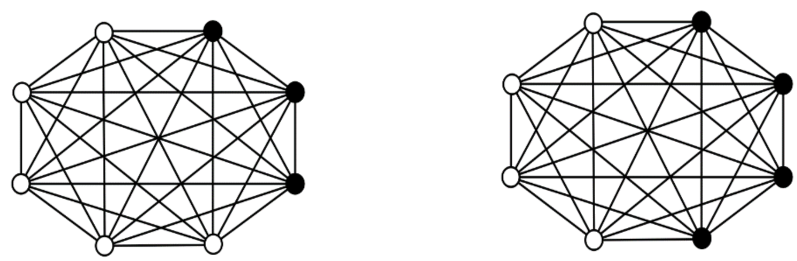
Proposition 2.2 For a complete graph $K_n (n \geq r+1)$ we have $\gamma_r (K_n) = n - r$.

Proof. A complete graph K_n of order $n, (n \geq 2)$, let the vertices of complete graph be $V(K_n) = \{ v_1, v_2, \ldots, v_n \}$. Let $D_r = D_1 \cup D_2 \cup D_3$, since $v_1, v_2, \ldots, v_{n-1}$ adjacent with v_n by one edge, the $D_1 = \{v_1, v_2, \ldots, v_{n-1}\}$ is minimum single dominating set. Since $v_1, v_2, \ldots, v_{n-2}$ adjacent with the set $\{v_n, v_{n-1}\}$ by two edges, then $D_2 = \{v_1, v_2, \ldots, v_{n-2}\}$ is minimum bi-dominating set. Since $v_1, v_2, \ldots, v_{n-3}$ adjacent with the set $\{v_n, v_{n-1}, v_{n-2}\}$ by three edges, the $D_3 = \{v_1, v_2, \ldots, v_{n-3}\}$ is minimum triple effect dominating set. Hence, $D_r = \{v_1, v_2, \ldots, v_{n-r}\}$ is r-dominating set, so every vertex in r-dominating set D_r dominates r vertices, then D_r contains all vertices of K_n unless r vertices. For example, see Figure 2.



a) When r = 5 b) When r = 4

Figure 2: A minimum r-dominating set in K_8.

Theorem 2.3 For a complete bipartite graph $K_{n,m}$, we have

$$\gamma_r (K_{n,m}) = \begin{cases} n & if \ m = r \ , n \geq 1 \\ n + m - 2r & if \ n, m > r \end{cases}$$

Proof. Let $\{ V_1, V_2 \}$ be a partition of the complete bipartite graph $K_{n,m}$ such that $V_1 = \{ v_1, v_2, \ldots, v_n \}$ and $V_2 = \{ u_1, u_2, \ldots, u_m \}$.

A. If $m = r$ and $n \geq 1$ as follows:

Case 1. $n = 1$ then $\gamma_r (K_{1,r}) = 1$. Hence, $D = \{v_1\}$ is minimum r-domination set.

Case 2. $n > 1$, let $\{ v_1, v_2, \ldots, v_n \}$ dominating on $\{ u_1, u_2, \ldots, u_r \}$ such that $D_r = |\{v_i\}_1^n| = n$ dominating set. Hence $\gamma_r (K_{n,r}) = n$.

B. If $n, m > r$, let $D_r = D_1 \cup D_2 \cup D_3$, since $v_1, v_2, \ldots, v_{n-1}$ adjacent with u_m by one edge and $u_1, u_2, \ldots, u_{m-1}$ adjacent with v_n by one edge then $D_1 = \{v_1, v_2, \ldots, v_{n-1}, u_1, u_2, \ldots, u_{m-1}\}$ is minimum single dominating set. since $v_1, v_2, \ldots, v_{n-2}$ adjacent with the set $\{u_m, u_{m-1}\}$ by

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two edge and $u_1, u_2, \dots, u_{m-2}$ adjacent with the set $\{v_n, v_{n-1}\}$ by two edge then $D_2 = \{v_1, v_2, \dots, v_{n-2}, u_1, u_2, \dots, u_{m-2}\}$ is minimum bi-dominating set. since $v_1, v_2, \dots, v_{n-3}$ adjacent with the set $\{u_m, u_{m-1}, u_{m-2}\}$ by three edge and $u_1, u_2, \dots, u_{m-3}$ adjacent with the set $\{v_n, v_{n-1}, v_{n-2}\}$ by three edge then $D_3 = \{v_1, v_2, \dots, v_{n-3}, u_1, u_2, \dots, u_{m-3}\}$ is minimum triple effect dominating set. Hence, $D_r = \{v_1, v_2, \dots, v_{n-r}, u_1, u_2, \dots, u_{m-r}\}$ is dominating set. Where all the $n-r$ vertices will dominate the *r* vertices of V_2. Also, all $m-r$ vertices of V_2 which are in D_r will dominate the *r* vertices of V_1, that belong to *V\D*. Hence, $\gamma_r\left(K_{n,m}\right) = n + m - 2r$. For example, see Figure 3.

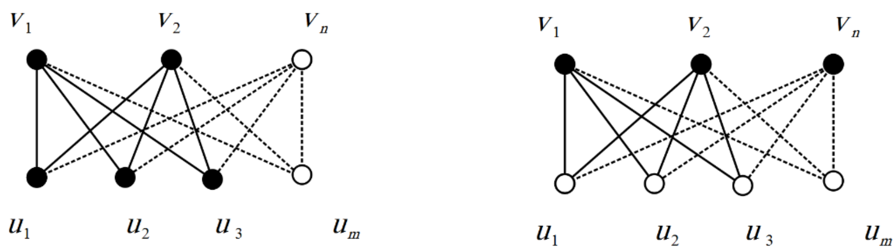

Figure 3: A minimum *r*-dominating set in $K_{n,m}$.

Proposition 2.4 Let *G* be a wheel graph W_n with $n+1$ vertices $(n \geq 3)$ then:

$$\gamma_r(W_n) = \begin{cases} n & if\ r = 1 \\ 1 & if\ r = n\ , r \geq 4 \end{cases}$$

Proof. By the definition of wheel graph there is a cycle C_n and complete graph K_1, let the vertices of this graph labeled by $V(W_n) = \{v_1, v_2, \dots, v_{n+1}\}$ such that $\deg(v_1, v_2, \dots, v_n) = 3$ and v_{n+1} is the vertex of degree *n*.

If $r = 1$, since every vertex in *D* dominates exactly one vertex from *V\D*. Then, *D* must be containing all vertices of W_n unless the vertex of K_1. Hence, $D_1 = \{v_1, v_2, \dots, v_n\}$ is minimum *r*-dominating set.

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If $n = r$, let v_{n+1} adjacent with $v_1, v_2, \dots, v_n$ by one edge. Hence, $D_r = \{v_{n+1}\}$ is minimum r-dominating set. If $n > r(n, r \geq 4)$ by definition of wheel, since $\deg(v_1, v_2, \dots, v_n) = 3$ then W_n has no r-dominating set. For example, see Figure 4.

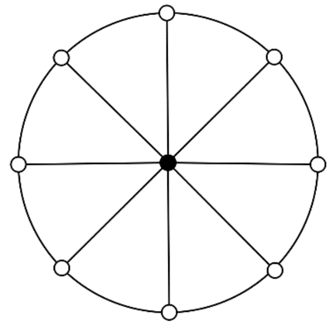

Figure 4: A minimum r-dominating set in W_8 .

Proposition 2.5 The lollipop graph $L_{m,n}$ has r-dominating set if and only if $m = r$ and $n = 1, r \geq 3$ where, $\gamma_r\left(L_{m,n}\right) = 1$.

Proof. By the definition of lollipop graph there is a complete graph K_m and path P_n, then $L_{m,1}$ has K_m and P_1, let $V\left(L_{m,1}\right) = \{v_1, v_2, \dots, v_{m+1}\}$ such that v_2 adjacent the vertex of a path. Hence, $D_r = \{v_2\}$ is minimum r-dominating set. If $m > r$ and $n > 1$, then $L_{m,1}$ has no r-dominating set.

Proposition 2.6 For the barbell graph $B_{n,n}$, we have $\gamma_r\left(B_{n,n}\right) = 2n - 2r$, $n \geq r + 1$.

Proof. By the definition of barbell graph is a graph formed by connecting two copies of a complete graph K_n by a bridge, let $V(B_{n,n}) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$, since $\gamma_r(K_n) = n - r$ according to proposition (2.2). Then, $D_r = \{v_1, v_2, \dots, v_{n-r}, u_1, u_2, \dots, u_{n-r}\}$ is dominating set. Hence, it's clear then $\gamma_r\left(B_{n,n}\right) = (n - r) + (n - r) = 2n - 2r$.

Theorem 2.7 Let P_n is a path graph then $\overline{P_n}$ has r-domination number if and only if $r + 3 \leq n \leq 2r + 3$ such that :

$$\gamma_r(\overline{P_n}) = \begin{cases} 2 & if\ n = r + 3, r + 4 \\ n - (r + 2) & if\ n = r + 5, r + 6, \dots, 2r + 3 \end{cases}$$

Where $\overline{P_n}$ has no r-domination set for $n < r + 3\ or\ n > 2r + 3$.

Proof. Since $\deg(v_i) \leq r - 1\ \ \forall\ v_i \in \overline{P_i}, i = 2, 3, \dots, r + 1,$ then $\overline{P_n}$ has no r-domination, where $n = r + 2$ if $D = \{v_1\}$, there is one vertex is not dominated by D .If $D = \{v_{r+2}\}$, there is one vertex

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is not dominated by D. If $D = \{v_1, v_{r+2}\}$. There every one of them dominates two vertices, all above cases are contraction our definition, so $\bar{P}_{r+2}$has no r-domination. If $n = r + 3$, either $D = \{v_1, v_{r+3}\}$ or $D = \{v_i, v_{i+1}\}; i = 2,3, \dots, r + 1$. If $n = r + 4$, then $D = \{v_i, v_j\}$ such that $d(v_i, v_j) = 2$ and $i, j \neq 1, r + 4$. If $n = r + 5, r + 6, \dots, 2r + 3$, let $D = \{v_{2i}, i = 1,2, \dots, n - (r + 2)\}$ every vertex in D dominates r vertices, in all above cases, D is a minimum r-dominating set. Thus, D is a $\gamma_r - set$ of $\bar{P_n}$.

If $n > 2r + 3$, then every dominating set D has at least one vertex dominates less then r vertices or dominates more than r vertices.

Theorem 2.8 Let C_n be a cycle graph of order $n \geq 3$, then $\overline{C_n}$ has r-domination number if and only if $r + 3 \leq n \leq 2r + 4$ and $n = 3(r + 1)$, such that :

$$\gamma_r(\overline{C_n}) = \begin{cases} 2 & if\ n = r + 3 \\ n - (r + 2) & if\ n = r + 4, r + 5, \dots, 2r + 4 \\ 2r + 2 & if\ n = 3(r + 1) \end{cases}$$

Proof. Since $\deg(v_i) \leq r - 1\ \forall\ v_i \in \overline{C_i}, i = 3, 4, \dots, r + 1$, then $\overline{C_n}$ has no r-domination. If $n = r + 3$, then D have any two consecutive vertices for $\bar{C}_{r+3}$. If $n = r + 4$ then $D = \{v_i, v_j\}$ such that $d(v_i, v_j) = 3$. If $n = r + 5, r + 6, \dots, 2r + 4$, let $D = \{v_{2i-1}, i = 1,2, \dots, n - (r + 2)\}$ then all vertices of D is adjacent together and dominate exactly r vertices. If $n = 3(r + 1)$, let $D = \{v_i, v_{i+1}, i = 1,4,7,10, \dots, n - 2\}$, then all vertices of D is adjacent together and dominate exactly r vertices. In all above cases. Hence D is a minimum r-dominating set. Thus, $D$is a $\gamma_r - set$ of $\overline{C_n}$.

If $2r + 5 \leq n \leq 3r + 2$ and $n > 3r + 4$, then every dominating set D has at least one vertex dominates less then r vertices or dominates more than r vertices.

Theorem 2.9 Let $K_{n,m}$ be a bipartite graph, then $\bar{K}_{n,m}$ has r-domination number if and only if $n > r$ and $m > r$ such that $\gamma_r(\bar{K}_{n,m}) = n + m - 2r$

Proof. The vertices for this graph are labeled by: $V(\bar{K}_{n,m})$=$\{v_i^j, i = 1, 2, 3, \dots, n\ , j = 1, 2, 3, \dots, m\}$. If $n > r$ and $m > r$ then $\bar{K}_{n,m}$ contains two graphs K_n and K_m , let D=$\{v_i^j, i = 1, 2, 3, \dots, n - r\ , j = 1, 2, 3, \dots, m - r\}$ then from proposition (2.2.) for a complete graph K_n and K_m $(n \geq r + 1)$, since $\gamma_r\ (K_n) = n - r$ and $\gamma_r\ (K_m) = m - r$ such that $\bar{K}_{n,m}$ and every

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graph of them has r-dominating. Hence, it's clear that $\gamma_r\left(\overline{K}_{n,m}\right) = (n-r) + (m-r) = n + m - 2r$.

3. Conclusions

In conclusion, this research has made significant contributions to the field of graph theory by expanding the concepts of bi-domination and triple effect domination to investigate novel type of domination is r −domination number. Our investigations have yielded valuable information about how r-domination behaves in different graph structures, including path, cycle, complete, complete bipartite, wheel, lollipop, and barbell graphs. Furthermore, we have extended our analysis to complement graphs, enriching our understanding of r −domination in various graph families.

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هيمنة r لبعض انواع البيانات الخاصة

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المستخلص

في هذا البحث ، قدمنا نوع جديد من الهيمنة في البيانات اسمها هيمنة r ، قمنا بتوسيع الهيمنة الثنائية وهيمنة التأثير الثلاثي في البيانات. لتكن G رسم بياني منتهي وبسيط . D مجموعة جزئية من مجموعة الرؤوس V فتكون مجموعة مهيمنة r في G اذا كان كل $D \ni u$ يهيمن على r من الرؤوس في $V\setminus D$. يرمز للحد الأدنى من المجموعة المهيمنة r بالرمز $\gamma_r(G)$. تمت دراسة هيمنة r لبعض انواع البيانات الخاصة .

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