An Efficient Three-step Iterative Methods Based on Bernstein Quadrature Formula for Solving Nonlinear Equations

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Abstract

In this study, we suggest and analyze two new one-parameter families of an efficient iterative methods free from higher derivatives for solving nonlinear equations based on Newton theorem of calculus and Bernstein quadrature formula, Bernoulli polynomial basis, Taylor's expansion and some numerical techniques. We prove that the new iterative methods reach orders of convergence ten with six and eight with four functional evaluations per iteration, which implies that the efficiency index of the new iterative methods is $(10)^{1/6} \cong 1.4678$ and $(8)^{1/4} \cong 1.6818$ respectively. Numerical examples are provided to show the efficiency and performance of our iterative methods, compare to Newton's method and other relevant methods.

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1. Introduction:

A frequently occurring and most important problem in mathematics, science and engineering is how to find the solution of nonlinear equations which can be expressed in general as follows

$$f(x) = 0, \tag{1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D.

Since the numerical analysis is to devise algorithms that give quick and accurate answers to mathematical problems for scientists and engineers, nowadays using computers. Therefore, numerically iterative methods are often the only choice for solving this general problem.

The Newton's method is one of the famous classical iterative methods to find the root of equation (1). The iterative scheme is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,\dots \tag{2}$$

which it has quadratic convergence, [1]. In the recent past, much attention has been given to developed several iterative methods for solving the nonlinear equations. Many of iterative methods have been obtained by using different techniques such as Taylor expansion, decomposition, homotopy, variational iteration, geometric methods and quadrature formulas also, we know that quadrature formula plays an important role in the evaluation of the numerical integrals.

The first study of quadrature formula was by S. Weerakoon and T.G.I. Fernando in 2000, studied new variant of Newton's method based on trapezoidal instead of a rectangle and they got new two-step iterative method. It has third-order convergence, [2], defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \quad n = 0, 1, \dots$$

By improving Newton's method say, V. I. Hasanov et al. in 2002, modified Newton's method by approximate the definite integral in quadrature rule by using Simpson's formula and they obtained a new two-step iterative method with third-order convergence, [3]. G. Nedzhibov in

2002, gave several classes of two-step iterative methods by using different quadrature rules, [4]. M. Frontini and E. Sormani in 2003, extended the results of the iterative methods in [2], and got new two- step iterative method of order three independent of the integration formula, [5]. A. Y. Ozban et al. in 2004, presented some new two-step variant of Newton's method based on harmonic mean and midpoint integration rule, [6]. H.H.H. Homeier in 2005, modified the iterative method in [2] by using Newton's theorem for the invers function and he got new classes of iterative methods with cubic convergence, [7]. M. A. Noor in 2007, suggested new two-step iterative methods also by using quadrature formula, [8]. L. Liu and X. Wang in 2010, proposed new three-step iteration scheme by using the method of weight functions, [9]. M. A. Noor et al. in 2010, suggested and analyzed some new iterative methods for solving the nonlinear equations using the decomposition technique coupled with the system of equations, [10]. X. Wang and L. Liu in 2010, derived two new three-step iterative methods based on Newton's method and modified Ostrowski's method with an eighth-order convergence for solving the simple roots of nonlinear equations by Hermite interpolation methods, [11]. A. Cordero and J.R. Torregrosa in 2011, Presented a new three-step family of eighth-order methods obtained an eight-order convergence based on Ostrowski's method, [12]. J. Jayakumar in 2013, proposed a generalization of two-step Simpson- Newton's method where Simpson's integration rule is applied for approximating the definite integral in quadrature formula, [13]. J. R. Sharma and H. Arora in 2014, presented a family of three-point iterative methods for solving nonlinear equations, [14]. O. Oghovese and E. O. John in 2014, introduced new two–step family of iterative method based on composite trapezoid rule and fundamental theorem of calculus, [15]. O. Oghovese and E. O. John in 2014, proposed a new three steps iterative method of order six for solving nonlinear equations, [16]. A.A. Al-Harbi and I.A. Al-Subaihi in 2015, a new family of three-step optimal eighth-order iterative methods are presented, [17]. M. Saqib and M. Iqbal in 2017, used quadrature rule to approximate the definite integral by rectangle integral rule and midpoint integral rule and they obtained new two-step iterative methods, [18]. R. Thukral in 2018, proposed new three-step Simpson's type method requires the same number of evaluations of the function as classical method but of fifth order convergence, [19]. U. k. Qureshi in 2019, Suggested a new iterative method of order two which is derived from quadrature formula by approximate the definite integral by using composite trapezoidal rule and some numerical techniques, [20]. G. Sana et al. in 2020, introduced two new three-step iterative schemes by applied quadrature formula and decomposition approach, [21]. B. Neta in 2021, developed a derivative-free method with memory based on Traub's method as the first step, [22]. C. Zalinescuin 2021, introduced several methods for comparing two convergent iterative processes for the same problem, [23]. G. Sana et al. in 2021, suggested and analyzed some new q-iterative methods by using the q-analogue of the Taylor's series and the coupled system technique, [24].

In this paper, we present new families of iterative methods for solving equation (1) by using Bernstein integration formula to approximate the definite integral in the quadrature rule and we find that some of well-known iterative methods can be deduced as special cases from the proposed iterative methods. We approximate the higher derivatives in the new three-step iterative methods to reduce the number of functions needed in each iteration to update the efficiency index. Also, we introduce some numerical examples that confirm the theoretical results allow us to compare these methods with Newton's method and with other relevant methods. Moreover, we introduce the graphical analysis for the uphold of numerical results.

2. Preliminaries

Offers some basic definitions, theorem and lemma that we need in our work.

Definition 2.1, [25]: A sequence of iterates $\{x_n\}$ is said to converge to the root $\alpha \in R$ if

$$\lim_{n\to\infty}|x_n-\alpha| = 0.$$

If $x_n, x_{n-1}, \dots, x_{n-m+1}$ are m approximates to a root, then we write an iteration method in the form

$$x_{n+1} = \varphi(x_n, x_{n-1}, \dots, x_{n-m+1}), \tag{3}$$

where we have written the equation (1) in the equivalent form

$$x = \varphi(x)$$

The function φ is called the iteration function. For $m = 1$, we get the one-point iteration method

$$x_{n+1} = \varphi(x_n), \quad n = 0,1,2,\dots \tag{4}$$

If $\varphi(x)$ is continuous in the interval $[a, b]$ that contains the root and $|\varphi'(x)| \leq c < 1$ in this interval, then for any choice of $x_0 \in [a, b]$, the sequence of iterates $\{x_n\}$ obtained from (4) converges to the root of $x = \varphi(x)$ or $f(x) = 0$.

Thus, for any iterative method of the form (3) or (4), we need the iteration function $\varphi(x)$ and one or more initial approximations to the root.

In practical applications, it is not always possible to find α exactly. We therefore attempt to obtain an approximate root x_{n+1} such that

$$|f(x_{n+1})| < \varepsilon \tag{5}$$

and / or

$$|x_{n+1} - x_n| < \varepsilon \tag{6}$$

where x_n and x_{n+1} are two consecutive iterates and ε is the prescribed error tolerance.

Definition 2.2, [2]: Let $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D with a simple root α of the nonlinear equation. An iterative method is said to have an integer order of convergence p if it produces the sequence $\{x_n\}$ of real numbers such that

$$\lim_{x\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = A \neq 0,$$

for some $A \neq 0$ and $p \geq 1$, then p is said to be the order of convergence of the sequence, and A is known as the asymptotic error constant.

or equivalently

$$x_{n+1} - \alpha = A(x_n - \alpha)^p + O((x_n - \alpha)^{p+1})$$

Notation 2.1, [2]: Let $e_n = x_n - \alpha$ is the error in the n$^{\text{th}}$ iteration. The equation

$e_{n+1} = ce_n^p + O\left(e_n^{p+1}\right)$ is called the error equation for the method, p being the order of convergence.

Definition 2.3, [13,2]: Let α be a root of the nonlinear equation and suppose that x_{n-1}, x_n *and* x_{n+1} are three successive iterations closer to the root α. Then, the computational order of convergence (COC) denoted by ρ can be computed using the formula

$$\rho = \frac{ln|e_{n+1}/e_n|}{ln|e_n/e_{n-1}|}.$$

Definition 2.4, [19]: The efficiency of a method is measured by the index

$$\mathrm{E.I} = p^{\frac{1}{\omega}},$$

where p is the order of convergence and ω is the total number of function evaluations per iteration.

Theorem 2.1, [17]: Let $\psi_1(x), \psi_2(x), \ldots, \psi_r(x)$ be iterative functions with the orders $s_1, s_2, \ldots, s_r$, respectively. Then the composition of iterative functions

$$\psi(x) = \psi_1(x)\big(\psi_2(x)(\ldots(\psi_r(x))\ldots)\big)$$

defines the iterative method of the order $s_1 s_2 \ldots s_r$.

Corollary 2.1, [26,27]: For a continuous function $f(x)$on [0, 1], we have

$$\int_a^b f'(x)dx \approx B_m(f', x) = \frac{b-a}{m+1}\sum_{k=0}^{m} f'\left(a + (b-a)\frac{k}{m}\right).$$

3. Construct of New Iterative Methods

In this section, we construct new Newton-type iterative methods and their modifications based on Newton's theorem of calculus and Bernstein quadrature formula.

Let $\alpha \in D$ be a simple root of equation (1) and x_0 is initial guess sufficiently close to α.

Consider Newton's theorem of calculus, defined by

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(\lambda)\, d\lambda \qquad (7)$$

If we approximate the definite integral in equation (7), by using Corollary 2.1, we have

$$f(x) = f(x_0) + \frac{(x-x_0)}{m+1}\sum_{k=0}^{m} f'\left(x_0 + (x-x_0)\frac{k}{m}\right) \qquad (8)$$

From equation (1) and solving equation (8) for x, we obtain

$$x = x_0 - \frac{(m+1)f(x_0)}{\sum_{k=0}^{m} f'\left(x_0 + (x - x_0)\frac{k}{m}\right)} \tag{9}$$

Since the equation (9) is implicit we can overcome this by approximate $(x - x_0)$ in the right-hand side by $\left(-\beta \frac{f(x_0)}{f'(x_0)}\right)$, we obtain

$$x = x_0 - \frac{(m+1)f(x_0)}{\sum_{k=0}^{m} f'\left(x_0 - \beta\frac{k}{m}\left(\frac{f(x_0)}{f'(x_0)}\right)\right)} \tag{10}$$

Also, we can get from equation (10) by using Taylor expansion of $f'\left(x_0 - \beta \frac{k}{m}\frac{f(x_0)}{f'(x_0)}\right)$, and neglecting the terms of the third order and above, we have

$$x = x_0 - \frac{(m+1)f(x_0)f'(x_0)}{\sum_{k=0}^{m}\left(\left(f'(x_0)\right)^2 - \beta\frac{k}{m}f(x_0)f''(x_0)\right)} \tag{11}$$

Now, by using equations (10) and (11), we can suggest the following new one-step, two-step and three-step one-parameter family of iterative methods for solving nonlinear equation (1), respectively.

Algorithm 2.1: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$x_{n+1} = x_n - \frac{(m+1)f(x_n)}{\sum_{k=0}^{m} f'\left(x_n - \beta\frac{k}{m}\frac{f(x_n)}{f'(x_n)}\right)}, \quad n = 0,1,\ldots$$

when $\beta = 0$ and $m = 1$, we get Newton's method.

Algorithm 2.2: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$x_{n+1} = x_n - \frac{(m+1)f(x_n)f'(x_n)}{\sum_{k=0}^{m}\left\{\left(f'(x_n)\right)^2 - \beta\frac{k}{m}f(x_n)f''(x_n)\right\}}, \quad n = 0, 1, \ldots$$

when $\beta = 0$ and $m = 1$, we get Newton's method and also, when $\beta = 1$ and $m = 1$, we get Halley's method in [28].

Algorithm 2.3: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \ ,$$

$$x_{n+1} = y_n - \frac{(m+1)f(y_n)}{\sum_{k=0}^{m} f'\left(y_n - \beta \frac{k\, f(y_n)}{m f'(y_n)}\right)} \ , \qquad n = 0, 1, \dots$$

Algorithm 2.4: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \ ,$$

$$x_{n+1} = y_n - \frac{(m+1)f(y_n)f'(y_n)}{\sum_{k=0}^{m}\left\{\left(f'(y_n)\right)^2 - \beta \frac{k}{m} f(y_n) f''(y_n)\right\}} , \qquad n = 0, 1, \dots$$

Algorithm 2.5: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \ ,$$

$$z_n = y_n - \frac{(m+1)f(y_n)}{\sum_{k=0}^{m} f'\left(y_n - \beta \frac{k\, f(y_n)}{m f'(y_n)}\right)} \ ,$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \ , \qquad n = 0, 1, \dots$$

Algorithm 2.6: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \ ,$$

$$z_n = y_n - \frac{(m+1)f(y_n)f'(y_n)}{\sum_{k=0}^{m}\left\{\left(f'(y_n)\right)^2 - \beta \frac{k}{m} f(y_n) f''(y_n)\right\}} \ ,$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \ , \qquad n = 0, 1, \dots$$

Since the purpose of our research is to obtain an efficient three-step iterative methods of higher order of convergence for solving equation (1), so we depend on Algorithms (3.5) and (3.6). Furthermore, to update the efficiency index of our iterative methods so, we approximate the derivatives $f'(y_n), f''(y_n)$ and $f'(z_n)$ respectively, to reduce the number of functional evaluations needed in each step of iteration by using an orthogonal polynomial as basis. This idea is very important and plays a significant part in developing many iterative methods. Now we look for an approximation of $f'(y_n), f''(y_n)$ and $f'(z_n)$ respectively.

Consider the function

$$Q(t) = \sum_{j=0}^{r} a_j\, B_j(t-\xi_n) \tag{12}$$

where $\xi_n \in \{x_n, y_n, z_n\}$, $a_j, j = 0, 1, 2, \dots, r$ are unknowns to be found, and

$B_j, j = 0, 1, 2, \dots, r$ are forms Bernoulli basis polynomial, defined by

$B_0(t-\xi_n) = 1,\ B_1(t-\xi_n) = (t-\xi_n) - \frac{1}{2}, B_2(t-\xi_n) = (t-\xi_n)^2 - (t-\xi_n) + \frac{1}{6}$ and

$B_3(t-\xi_n) = (t-\xi_n)^3 - \frac{3}{2}(t-\xi_n)^2 + \frac{1}{2}(t-\xi_n), \xi_n \in \{x_n, y_n, z_n\}$.

To approximate $f'(y_n)$ we construct a Bernoulli interpolation polynomial, that meets the interpolation conditions

$f(x_n) = Q(x_n), f'(x_n) = Q'(x_n)$ and $f(y_n) = Q(y_n)$.

Here, take $r = 2$ and $\xi_n = y_n$ and from equation (12), then $Q(t)$ can be written as:

$Q(t) = a_0 B_0(t-y_n) + a_1 B_1(t-y_n) + a_2 B_2(t-y_n)$.

Applying the interpolation conditions above on $Q(t)$, we get

$$f(x_n) = a_0 + a_1\left((x_n - y_n) - \frac{1}{2}\right) + a_2\left((x_n - y_n)^2 - (x_n - y_n) + \frac{1}{6}\right),$$

$$f(y_n) = a_0 - \frac{1}{2}a_1 + \frac{1}{6}a_2,$$

$$f'(x_n) = a_1 + a_2(2(x_n - y_n) - 1),$$

Solving the system above of three linear equations of three unknowns, we obtain

$$a_0 = f(x_n) - \left((x_n - y_n) - \tfrac{1}{2}\right)a_1 - \left((x_n - y_n)^2 - (x_n - y_n) + \tfrac{1}{6}\right)a_2,$$

$$a_1 = \left(\tfrac{2}{x_n - y_n} - \tfrac{1}{(x_n - y_n)^2}\right)f(x_n) + \left(\tfrac{1}{(x_n - y_n)^2} - \tfrac{2}{x_n - y_n}\right)f(y_n) + \left(\tfrac{1}{x_n - y_n} - 1\right)f'(x_n), \text{ and}$$

$$a_2 = \frac{1}{(x_n - y_n)^2}\left(f(y_n) - f(x_n)\right) + \frac{1}{x_n - y_n}f'(x_n).$$

After substituting the values of a_1 and a_2 in equation $f'(y_n) = a_1 - a_2$, we get

$$f'(y_n) = \tfrac{2}{x_n - y_n}\left(f(x_n) - f(y_n)\right) - f'(x_n) := H_1(x_n, y_n) \quad (13)$$

Also, to approximate $f''(y_n)$ we construct a Bernoulli interpolation polynomial, that meets the interpolation conditions

$$f(x_n) = Q(x_n), f(y_n) = Q(y_n) \; and \; f'(y_n) = Q'(y_n) \,.$$

Take $r = 2$ and $\xi_n = y_n$ *and* from equation (12), then $Q(t)$ can be written as:

$$Q(t) = a_0 B_0(t - y_n) + a_1 B_1(t - y_n) + a_2 B_2(t - y_n).$$

Again, applying the interpolation conditions above on $Q(t)$, we get

$$f(x_n) = a_0 + a_1\left((x_n - y_n) - \tfrac{1}{2}\right) + a_2\left((x_n - y_n)^2 - (x_n - y_n) + \tfrac{1}{6}\right),$$

$$f(y_n) = a_0 - \tfrac{1}{2}a_1 + \tfrac{1}{6}a_2,$$

$$f'(y_n) = a_1 - a_2.$$

Then by solving the system above of three linear equations of three unknowns, we obtain

$$a_0 = f(x_n) - \left((x_n - y_n) - \frac{1}{2}\right)a_1 - \left((x_n - y_n)^2 - (x_n - y_n) + \frac{1}{6}\right)a_2,$$

$$a_1 = -\tfrac{1}{(x_n - y_n)}\left(f(y_n) - f(x_n)\right) + (1 - (x_n - y_n))a_2, \text{ and}$$

$$a_2 = \tfrac{1}{(x_n - y_n)^2}f(x_n) - \tfrac{1}{(x_n - y_n)^2}f(y_n) - \tfrac{1}{(x_n - y_n)}f'(y_n).$$

After substituting the values of a_2 in equation $f''(y_n) = 2a_2$, we get

$$f''(y_n) = \frac{2}{(x_n-y_n)^2}\left(f(x_n) - f(y_n)\right) - \frac{2}{(x_n-y_n)}H_1(x_n, y_n) \ := H_2(x_n, y_n)\ . \quad (14)$$

Finally, to approximate $f'(z_n)$ we construct a Bernoulli interpolation polynomial, that meets the interpolation conditions

$$f(x_n) = Q(x_n), f(y_n) = Q(y_n), f(z_n) = Q(z_n)\ and\ f'(x_n) = Q'(x_n).$$

And take, $r = 3$ and $\xi_n = z_n$ and from equation (12), then $Q(t)$ can be written as:

$$Q(t) = a_0B_0(t - z_n) + a_1B_1(t - z_n) + a_2B_2(t - z_n) + a_3B_3(t - z_n).$$

Also, by applying the interpolation conditions above on equation $Q(t)$, we get

$$f(x_n) = a_0 + a_1\left((x_n - z_n) - \frac{1}{2}\right) + a_2\left((x_n - z_n)^2 - (x_n - z_n) + \frac{1}{6}\right) + a_3\left((x_n - z_n)^3 - \frac{3}{2}(x_n - z_n)^2 + \frac{1}{2}(x_n - z_n)\right),$$

$$f(y_n) = a_0 + a_1\left((y_n - z_n) - \frac{1}{2}\right) + a_2\left((y_n - z_n)^2 - (y_n - z_n) + \frac{1}{6}\right) + a_3\left((y_n - z_n)^3 - \frac{3}{2}(y_n - z_n)^2 + \frac{1}{2}(y_n - z_n)\right),$$

$$f(z_n) = a_0 - \frac{1}{2}a_1 + \frac{1}{6}a_2,$$

$$f'(x_n) = a_1 + a_2(2(x_n - z_n) - 1) + a_3\left(3(x_n - z_n)^2 - 3(x_n - z_n) + \frac{1}{2}\right).$$

Solving the system above of four linear equations of four unknowns, we obtain

$$a_0 = f(x_n) - \left(a - \frac{1}{2}\right)a_1 - \left(a^2 - a + \frac{1}{6}\right)a_2 - \left(a^3 - \frac{3}{2}a^2 + \frac{1}{2}a\right)a_3,$$

$$a_1 = \frac{1}{b-a}\left(f(y_n) - f(x_n)\right) - (b + a - 1)a_2 - \left(b^2 + ab + a^2 - \frac{3}{2}(b + a) + \frac{1}{2}\right)a_3,$$

$$a_2 = \frac{1}{ab}(f(z_n) - f(x_n)) + \frac{1}{b(b-a)}(f(y_n) - f(x_n)) - \left(b + a - \frac{3}{2}\right)a_3, \text{ and}$$

$$a_3 = -\frac{1}{a(b-a)}f'(x_n) - \frac{1}{a^2b}\left(f(z_n) - f(x_n)\right) + \frac{1}{b(b-a)^2}\left(f(y_n) - f(x_n)\right),$$

where $(x_n - z_n) = a$ and $(y_n - z_n) = b$.

After substituting the values of a_1, a_2 and a_3 in equation $f'(z_n) = a_1 - a_2 + \frac{1}{2}a_3$, we get

$$f'(z_n) = 2\left(\frac{f(x_n)-f(z_n)}{x_n-z_n} - \frac{f(y_n)-f(x_n)}{y_n-x_n}\right) + \frac{f(y_n)-f(z_n)}{y_n-z_n} + \frac{y_n-z_n}{y_n-x_n}\left(\frac{f(y_n)-f(x_n)}{y_n-x_n} - f'(x_n)\right) := H_3(x_n, y_n, z_n) \tag{15}$$

Therefore, we suggest new three-step one-parameter families of iterative methods free from second derivative for solving nonlinear equation (1) as follows:

Algorithm 3.7: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{(m+1)f(y_n)}{\sum_{k=0}^{m} f'\left(y_n - \beta\frac{k}{m}\left(\frac{f(y_n)}{H_1(x_n,y_n)}\right)\right)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{H_3(x_n,y_n,z_n)}, \quad n = 0, 1, \ldots$$

Algorithm 3.8: For a given x_0, compute the approximate solution x_{n+1} by the following iterative method

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{(m+1)f(y_n)H_1(x_n,y_n)}{\sum_{k=0}^{m}\{(H_1(x_n,y_n))^2 - \beta\frac{k}{m}f(y_n)H_2(x_n,y_n)\}},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{H_3(x_n,y_n,z_n)}, \quad n = 0, 1, \ldots$$

4. Analysis of Convergence

In the following Theorems, we establish the convergence of the present Algorithms (3.7) and (3.8) respectively, when $m = 1$.

Theorem 3.1: Let $\alpha \in D$, be a simple root of a sufficiently differentiable function $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. If x_0 is sufficiently close to α, then the Algorithm 3.7 has tenth-order convergence when β=1, while its convergence of eighth order for any $\beta \in \mathbb{R} - \{1\}$.

Proof: Let α be a simple root of $f(x) = 0$. (Since, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$).

Expanding $f(x_n)$and $f'(x_n)$ by using Taylor expansion about α, we get

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)] \tag{16}$$

Where $c_s = \frac{f^{(s)}(\alpha)}{s!f'(\alpha)}$, $s = 2,3, \ldots$ & $e_n = x_n - \alpha$. From (23), we have

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)] \tag{17}$$

Dividing equation (16) by (17), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-3c_4 + 7c_2 c_3 - 4c_2^3)e_n^4 + \ldots. \tag{18}$$

Also, we need to compute

$$y_n = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + \ldots. \tag{19}$$

Expanding $f(y_n)\ and\ f'(y_n)$ about α and using (19) we have

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2 c_3 + 5c_2^3)e_n^4 + \ldots] \tag{20}$$

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_2^3)e_n^3 + (6c_2 c_4 - 11c_2^2 c_3 + 8c_2^4)e_n^4 + \ldots] \tag{21}$$

$$H_1(x_n, y_n) = f'(\alpha)[1 + (2c_2^2 - c_3)e_n^2 + (6c_2 c_3 - 2c_4 - 4c_2^3)e_n^3 + \ldots] \tag{22}$$

$$f'\left(y_n - \beta \frac{f(y_n)}{H_1(x_n, y_n)}\right) = f'(\alpha)[1 - 2c_2^2(\beta - 1)e_n^2 + 4c_2(c_2^2 - c_3)(\beta - 1)e_n^3 + \ldots] \tag{23}$$

$$z_n = \alpha - c_2^3(\beta - 1)e_n^4 + 4c_2^2(c_2^2 - c_3)(\beta - 1)e_n^5 + \ldots \tag{24}$$

Expanding $f(z_n)$ about α and using (24) we have

$$f(z_n) = f'(\alpha)[-c_2^3(\beta - 1)e_n^4 + 4c_2^2(c_2^2 - c_3)(\beta - 1)e_n^5 + \ldots] \tag{25}$$

$$H_3(x_n,y_n,z_n) = f'(\alpha)[1 + (-2\beta c_2^4 + 2c_2^4 + c_2c_4)e_n^4 + (8(\beta-1)c_2^5 - 8c_3c_2^3(\beta-1) - 2c_2^2c_4 + 2c_2c_5 + 2c_3c_4)e_n^5 + \ldots] \quad (26)$$

Dividing equation (25) by (26), we get

$$\frac{f(z_n)}{H_3(x_n,y_n,z_n)} = f'(\alpha)[-c_2^3(\beta-1)e_n^4 + 4c_2^2(c_2^2 - c_3)(\beta-1)e_n^5 + \ldots] \quad (27)$$

From equations (18), (20), (21), (23) and (27) we obtain

$$x_{n+1} = \alpha + c_2^4(\beta-1)(c_2^3(\beta-1) - c_4)e_n^8 - 8c_2^3(\beta-1)\Big((\beta-1)c_2^5 - c_3c_2^3(\beta-1) - \left(\frac{3}{4}\right)c_2^2c_4 + \left(\frac{1}{4}\right)c_2c_5 + \left(\frac{3}{4}\right)c_3c_4\Big)e_n^9 + 2c_2^4\Big((\beta^3 + 15\beta^2 - 34\beta + 18)c_2^7 - \left(\frac{3}{2}\right)c3\left(\beta^2 + \left(\frac{68}{3}\right)\beta - 24\right)(\beta-1)c_2^5 + \left(5\left(\beta^2 - \left(\frac{21}{5}\right)\beta + \frac{33}{10}\right)\right)c_4c_2^4 + \left(12(\beta-1)\right)\left(c_3^2\beta - c_3^2 + \left(\frac{11}{24}\right)c_5\right)c_2^3 + \left(\left(\frac{3}{4}\right)\beta^2c_3c_4 + \left(-\left(\frac{3}{2}\right)c_6 + \left(\frac{41}{2}\right)c_3c_4\right)\beta - \left(\frac{43}{2}\right)c_3c_4 + \left(\frac{3}{2}\right)c_6\right)c_2^2 - \left(6(\beta-1)\right)\left(c_3c_5 + \left(\frac{3}{4}\right)c_4^2\right)c_2 - 6c_3^2c_4(\beta-1)\Big)e_n^{10} + O(e_n^{11}) \quad (28)$$

Implying that

$$e_{n+1} = c_2^4(\beta-1)(c_2^3(\beta-1) - c_4)\, e_n^8 + \cdots + O(e_n^{11}) \quad (29)$$

When $\beta = 1$ we have

$$e_{n+1} = c_2^4\left(c_4c_2^4 - \frac{1}{2}c_3c_4c_2^2\right)e_n^{10} + O(e_n^{11}) \quad (30)$$

Hence, Algorithm 3.7 has at least tenth-order convergence.

Theorem 3.2: Let $\alpha \in D$, be a simple root of a sufficiently differentiable function $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. If x_0 is sufficiently close to α, then the Algorithm 3.8 has at least eighth order of convergence for any $\beta \in \mathbb{R}$.

Proof: With the same assumptions of the previous theorem, we have

$$H_2(x_n,y_n) = 2c_2 + 4c_3e_n + (2c_2c_3 + 6c_4)e_n^2 + \ldots \quad (31)$$

$$z_n = \alpha - (c_2^2(\beta-1) + c_3)c_2e_n^4 + ((4\beta-4)c_2^4 + (-6\beta+8)c_2^2c_3 - 2c_2c_4 - 2c_3^2)e_n^5 + \ldots \quad (32)$$

Expanding $f(z_n)$ about α and using (29) we have

$$f(z_n) = f'(\alpha)[-(c_2^2(\beta-1)+c_3)c_2e_n^4 + ((4\beta-4)c_2^4 + (-6\beta+8)c_2^2c_3 - 2c_2c_4 - 2c_3^2)e_n^5+\dots] \quad (33)$$

$$H_3(x_n,y_n,z_n) = f'(\alpha)\left[1 - 2c_2((\beta-1)c_2^3 + c_2c_3 - \tfrac{1}{2}c_4)e_n^4 + (8(\beta-1)c_2^5 + (-12\beta+16)c_3c_2^3 - 6c_2^2c_4 + (-4c_3^2+2c_5)c_2 + 2c_3c_4)e_n^5+\dots\right] \quad (34)$$

Dividing equation (30) by (31), we get

$$\frac{f(z_n)}{H_3(x_n,y_n,z_n)} = f'(\alpha)[-c_2((\beta-1)c_2^2 + c_3)e_n^4 + (4(\beta-1)c_2^4 + (-6\beta+8)c_3c_2^2 - 2c_2c_4 - 2c_3^2)e_n^5+\dots] \quad (35)$$

Substituting equations (18), (20), (22), (31) and (35) in Algorithm 3.8 we get

$$x_{n+1} = \alpha + \left((\beta-1)c_2^2 + c_3\right)\left((\beta-1)c_2^3 + c_2c_3 - c_4\right)c_2^2e_n^8 + O(e_n^9) \quad (36)$$

Implying that

$$e_{n+1} = \left((\beta-1)c_2^2 + c_3\right)\left((\beta-1)c_2^3 + c_2c_3 - c_4\right)c_2^2e_n^8 + O(e_n^9) \quad (37)$$

When $\beta = 1$ we have

$$e_{n+1} = (c_2c_3 - c_4)c_3c_2^2e_n^8 + O(e_n^9)$$
(38)

Hence, Algorithm 3.8 has at least eighth-order convergence.

5. Numerical Examples

In this section, we apply new three-step iterative methods that defined in Algorithms (3.7) and (3.8), to solve several nonlinear equations and make the comparison of newly established iteration methods with classical Newton's method [1], S. Weerakoon et al method [2], and with some existing optimal eighth order methods. For example, R. Thukral method [29], L. Liu et al. method [9], X. Wang et al. methods [11], A. Cordero et al. method [12] and one of the methods by A.A. Al-Harbi [17]. The methods are given as follows:

Newton's method (NM) :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \text{ , } n = 0, 1, \dots$$

S. Weerakoon et al. method (WFM) :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \quad n = 0, 1, \dots$$

R. Thukral method (TM) :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{(f(x_n))^2 + (f(y_n))^2}{(f(x_n) - f(y_n))f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{1+\mu_i^2}{1-\mu_i}\right)^2 - 2(\mu_i)^2 - 6(\mu_i)^3 + \frac{f(z_n)}{f(y_n)} + \frac{4f(z_n)}{f(x_n)}\right], \quad n = 0, 1, \dots$$

where $\mu_i = \frac{f(y_n)}{f'(x_n)}$.

L. Liu et al. method (LWM) :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f'(x_n)}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right)^2 + \frac{f(z_n)}{f(y_n) - \mu f(z_n)} + \frac{4f(z_n)}{f(x_n) + \beta f(z_n)}\right], \quad n = 0, 1, \dots$$

where $\beta = \mu = 1$.

X. Wang et al. methods (BM8 and BM8-2) :

(BM8):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{f'(y_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n = 0,1, \dots$$

(BM8-2):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = y_n - \frac{f(y_n)}{2f[x_n,y_n] - f'(x_n)},$$

$$x_{n+1} = z_n - \frac{f(z_n)}{2f[x_n,z_n] + f[y_n,z_n] - 2f[x_n,y_n] + (y_n - z_n)f[y_n,x_n,x_n]}, \quad n = 0, 1, \dots$$

A. Cordero et al. method (CTM) :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = v_n - \frac{f(z_n)}{f'(x_n)} \frac{3(\beta_2 + \beta_3)(v_n - z_n)}{\beta_1(v_n - z_n) + \beta_2(y_n - x_n) + \beta_3(z_n - x_n)}, \quad n = 0, 1, \dots$$

where $\beta_1 = 0,\ \beta_2 = 1\ and\ \beta_3 = 0$ and

$$v_n = z_n - \frac{f(z_n)}{f'(x_n)} \left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \frac{1}{2} \frac{f(z_n)}{f(y_n) - 2f(z_n)} \right)^2 \right].$$

A. A. Al-Harbi et al. method (ASM) :

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)} \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = z_n - \{(1 + t_1^2 + 2t_1^3 + \mu t_1^4) + (-1 + \beta t_2) + (1 + 2t_3 + \gamma t_3^2)\} \frac{f(z_n)}{f[y_n,z_n]},$$

$n = 0, 1, \ldots$

where $\mu = 1, \beta = 0$ and $\gamma = -2$ and $t_1 = \frac{f(y_n)}{f'(x_n)}, t_2 = \frac{f(z_n)}{f'(y_n)}$ and $t_3 = \frac{f(z_n)}{f'(x_n)}$.

For writing programs, we use Maple 2016 program with 1000-digit floating point arithmetic (Digits: = 1000). We use the stopping criteria $|x_{n+1} - x_n| < \varepsilon$ and $|f(x_{n+1})| < \varepsilon$, where $\varepsilon = 10^{-15}$, for computer programs. Different test functions and their approximate root α found up to the 28th decimal places are given in Table 1. Table 2 shows a comparison between the various iterative methods depending on the number of iterations (IT), the values of $|x_{n+1} - x_n|$ and $|f(x_{n+1})|$ and computational order of convergence (COC). Figures (2.1- 2.8) show the graphical analysis for the uphold of numerical results.

Table 1: The test functions and their root α

	$\boldsymbol{f(x)}$	$\boldsymbol{\alpha}$
1	$f_1(x) = x^3 + 4x^2 - 15$	1.6319808055660635175221064455
2	$f_2(x) = (x-1)^3 - 1$	2.0000000000000000000000000000
3	$f_3(x) = x^3 - e^{-x}$	0.7728829591492101128487486048
4	$f_4(x) = (1 + cos\, x)(e^x - 2)$	0.6931471805599453095377829940
5	$f_5(x) = sin\, x - \frac{x}{2}$	1.8954942670339809471440357380
6	$f_6(x) = sin^{-1}(x^2 - 1) - \frac{x}{2} + 1$	0.5948109683983691775226562351
7	$f_7(x) = (sin\, x)^2 - x^2 + 1$	2.0000019101432763850749104202

Table 2. Comparison our iterative methods with Newton's method and relevant various methods

$f(x)$	x_0	Methods	IT	$\lvert x_{n+1}-x_n \rvert$	$\lvert f(x_{n+1}) \rvert$	COC
f_1	1.9	NM	5	4.7627717e-16	2.0179551e-30	2.0000070
		WFM	4	3.5538378e-26	1.9121725e-76	2.9997779
		TM	3	7.11868510e-22	6.7158442e-106	4.8688477
		LWM	3	1.2777913e-56	2.2612305e-448	7.9411776
		BM8	3	3.8860695e-63	2.6388788e-501	7.9592052
		BM8-2	3	2.1273196e-65	1.1467258e-519	7.9633384
		CTM	3	7.8460648e-62	1.0680800e-490	7.9548448
		ASM	3	7.71083310e-59	2.2975939e-466	7.9419739
		NBM	3	4.8981129e-143	0.0	11.9628371
		NBM1	3	1.4736548e-72	7.9792114e-578	7.9792114
f_2	1.8	NM	6	3.0908727e-21	2.8660481e-41	2.0000004
		WFM	4	4.8567907e-17	4.0097399e-49	3.0037564
		TM	4	1.2301362e-47	4.5069838e-234	5.0041787
		LWM	3	4.8376766e-34	4.5997183e-266	8.1771117
		BM8	3	9.5534938e-42	2.0817044e-328	8.1048138
		BM8-2	3	2.5437487e-45	2.3373966e-357	8.08560810
		CTM	3	2.8011765e-40	1.3478179e-316	8.1326889
		ASM	3	2.0337759e-35	2.7318796e-277	8.1892898
		NBM	3	1.3599031e-95	0.0	12.0908695
		NBM1	3	4.4756985e-53	5.3674172e-420	8.0212447
f_3	3.5	NM	9	6.5470578e-20	8.9491765e-39	2.0000016
		WFM	6	1.9993598e-16	1.9761660e-47	2.9966706

		TM	5	5.6254001e-47	4.5689177e-231	4.99626010
		LWM	4	4.4125419e-29	3.9927632e-227	7.7982095
		BM8	4	3.97083810e-39	8.1562120e-308	7.8800229
		BM8-2	4	7.1398043e-47	1.7077867e-370	7.9632156
		CTM	4	7.1286254e-40	3.7426175e-314	7.9329018
		ASM	4	1.4220098e-33	3.6783273e-263	7.8247620
		NBM	4	3.7317121e-125	0.0	9.8812164
		NBM1	4	1.8366324e-95	5.3992020e-759	7.9967565
f_4	0.9	NM	5	1.9230803e-29	1.8170046e-58	1.9999999
		WFM	4	1.8175039e-31	2.0473423e-93	2.9999130
		TM	3	1.8803857e-36	9.27421034e-181	4.7726412
		LWM	3	5.4674051e-82	5.2221645e-653	7.8238898
		BM8	3	3.6612034e-118	1.13663377e-945	7.8148441
		BM8-2	3	5.0785010e-89	2.4383002e-710	7.8705503
		CTM	3	2.4828066e-72	1.1379207e-575	7.9379761
		ASM	3	6.8395375e-89	6.3437628e-709	7.8293396
		NBM	3	1.2206835e-145	0.0	9.8772847
		NBM1	3	5.1313716e-89	2.6481682e-710	7.8706139
f_5	2.0	NM	5	1.7664965e-20	1.4787271e-40	1.9999997
		WFM	4	6.9218142e-35	8.2109603e-104	2.9999649
		TM	3	1.1448151e-27	1.1529071e-135	4.9292808
		LWM	3	1.0543348e-69	4.6254912e-553	7.9618721
		BM8	3	1.8860406e-80	2.8462341e-640	7.9718217
		BM8-2	3	7.4215447e-80	1.8459664e-635	7.9733717

		CTM	3	1.0683158e-75	1.0583247e-601	7.9677030
		ASM	3	8.5875488e-73	4.0972561e-578	7.9630357
		NBM	3	5.6305868e-131	4.0e-1000	9.9793032
		NBM1	3	1.5961638e-100	7.9898120e-803	7.9512710
f_6	0.3	NM	5	2.3916069e-19	1.6107744e-38	2.00000910
		WFM	4	1.5267255e-25	7.2841117e-76	2.9997002
		TM	3	7.0798766e-27	8.7741676e-134	5.05203810
		LWM	3	1.3240055e-59	5.6416104e-474	7.9104307
		BM8	3	6.1558807e-77	2.0560733e-614	7.9112957
		BM8-2	3	6.6289185e-66	5.7572363e-525	7.9018207
		CTM	3	5.3278165e-61	1.9435254e-485	7.9366747
		ASM	3	5.0701651e-72	9.5018919e-575	7.9343293
		NBM	3	3.5116305e-108	3.0e-1000	9.9351292
		NBM1	3	1.1274703e-65	4.7650905e-523	7.8942633
f_7	1.5	NM	5	4.4131593e-19	3.7884542e-37	2.0000007
		WFM	4	1.8211104e-31	9.8628863e-93	2.9999346
		TM	3	9.9397361e-26	5.1884889e-125	4.9257651
		LWM	3	2.3530759e-66	3.5837479e-525	7.9648201
		BM8	3	1.8248762e-74	5.5373256e-591	7.9724181
		BM8-2	3	6.3807720e-77	6.2926145e-611	7.9760834
		CTM	3	3.8163135e-72	3.6529135e-572	7.9717647
		ASM	3	5.14822910e-69	9.8364499e-547	7.9641333
		NBM	3	6.4745225e-122	0.0	9.9818428
		NBM1	3	2.2780807e-86	1.8964177e-687	7.9659951

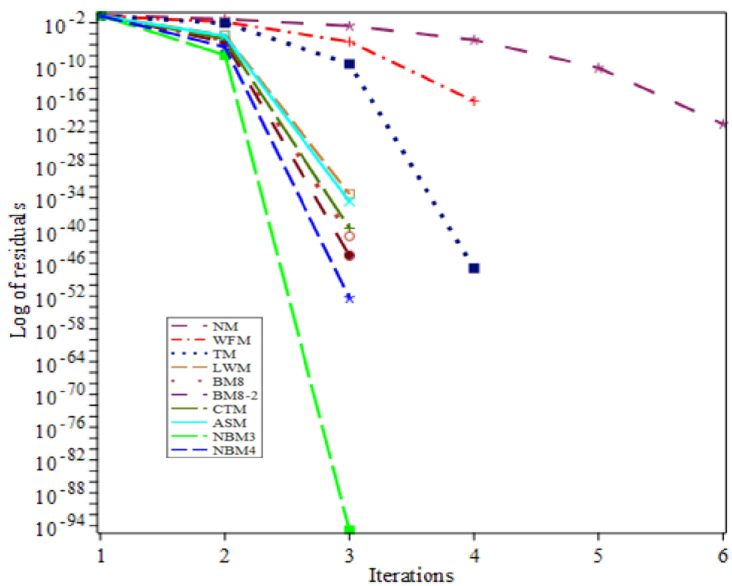

Figure 1: Log of residuals of problem 1

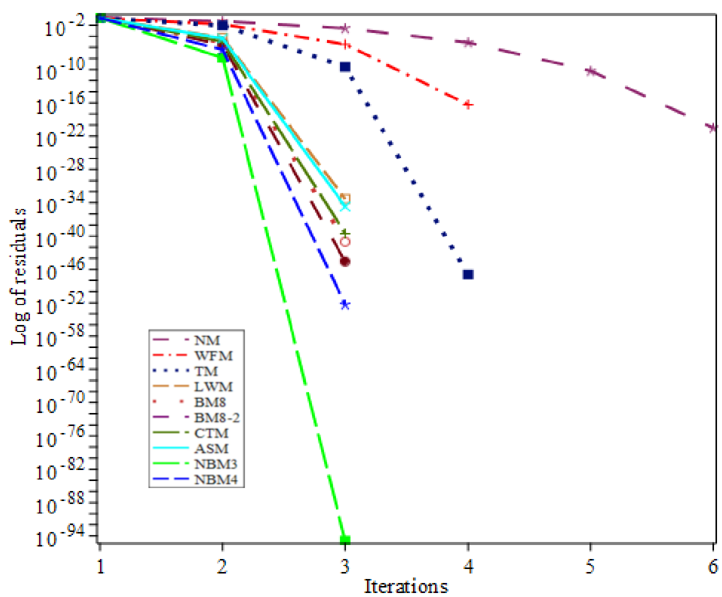

Figure 2: Log of residuals of problem 2

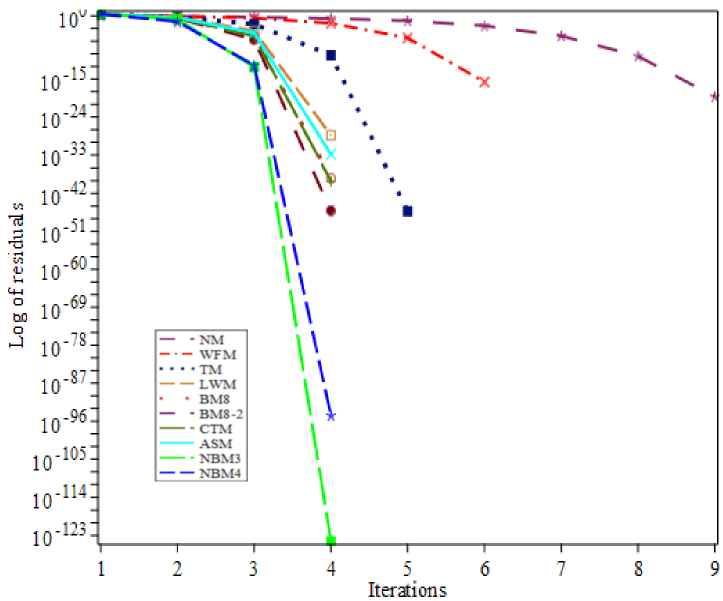

Figure 3: Log of residuals of problem 3

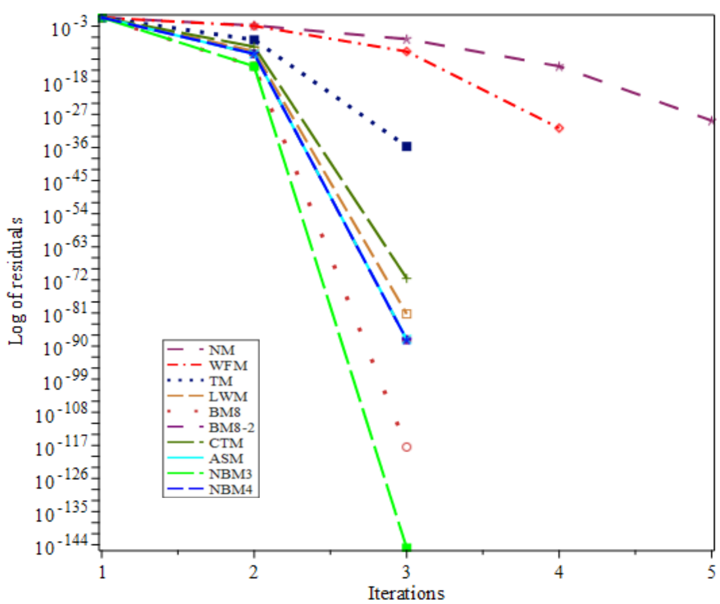

Figure 4: Log of residuals of problem 4

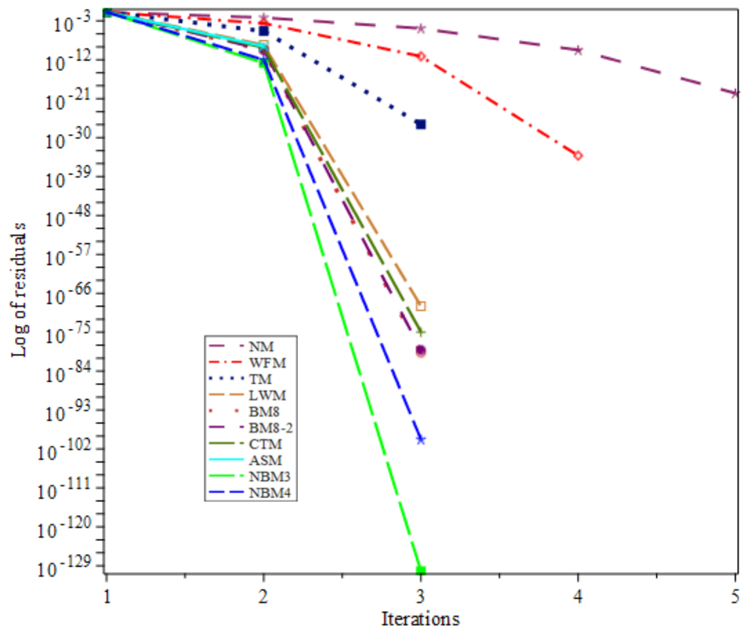

Figure 5: Log of residuals of problem 5

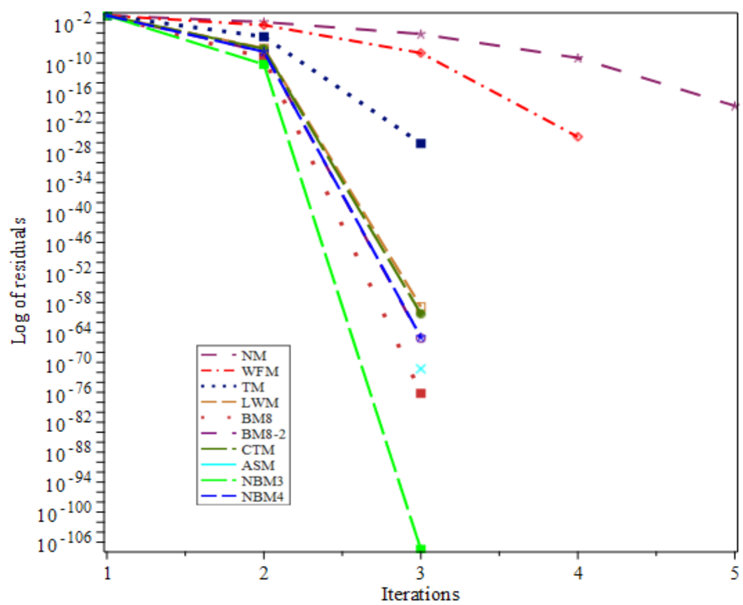

Figure 6: Log of residuals of problem 6

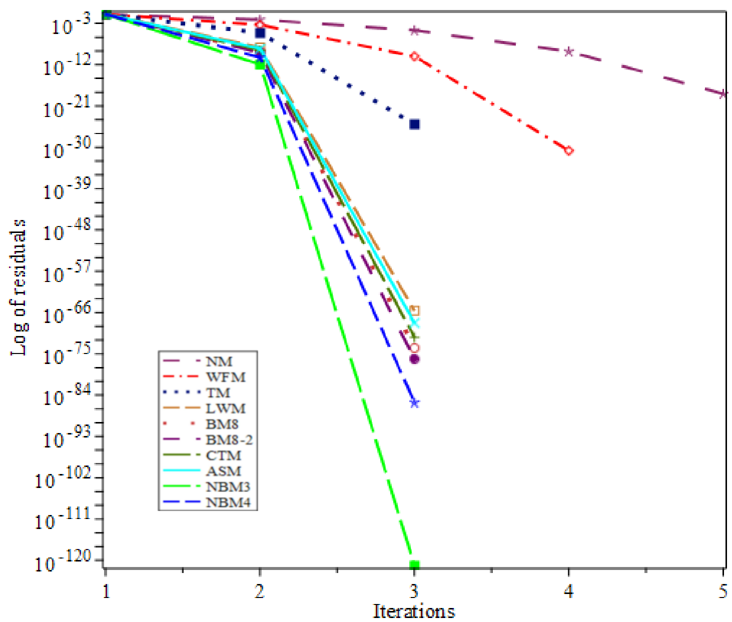

Figure 7: Log of residuals of problem 7

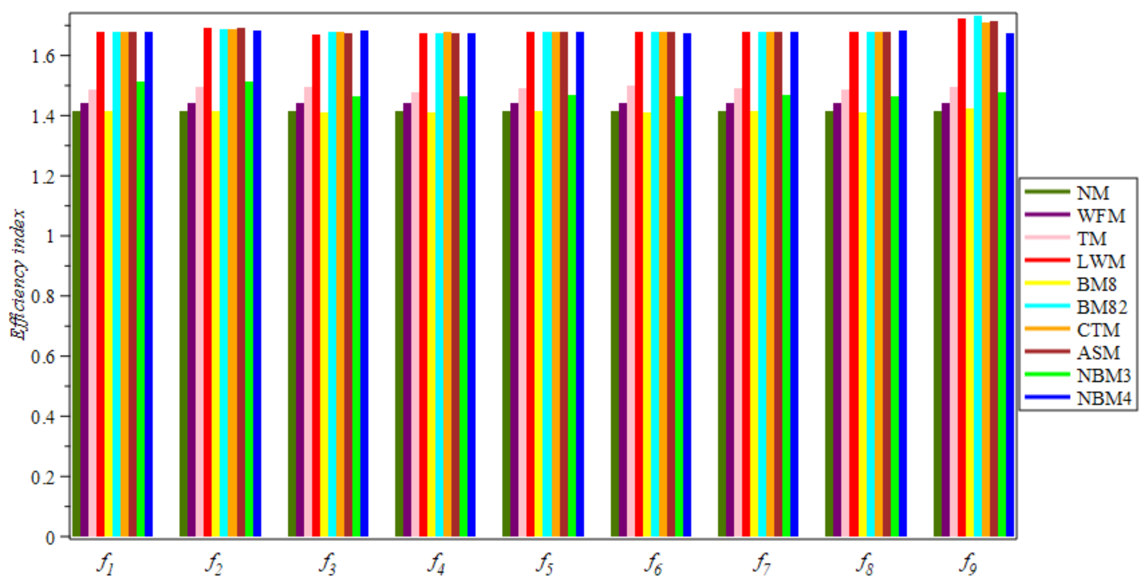

Figure 8: Comparison between methods and efficiency indices

5. Conclusions

In this research, new predictor-corrector iterative methods have been proposed for solving nonlinear equations denoted by (NBM) and (NBM1), respectively. Our new iterative methods have the advantage of evaluating only the first derivative of $f(x)$. Numerical results that we got show the convergence order of (NBM) and (NBM1) methods is ten and eight respectively, which is higher than many existing methods. Also, the number of iterations of (NBM) and (NBM1) methods is better than the classical Newton's method, S. Weerakoon et al. method and equal with other existing methods. The efficiency index of both new iterative methods is much better from the classical Newton's method, S. Weerakoon et al. method and (BM8) method and the efficiency index of (NBM1) method is equal with all the existing methods but, the efficiency index of (NBM) method is less. However, the drawback in the efficiency index of (NBM) method is compensated by increase in accuracy. Moreover, the proposed (NBM) and (NBM1) methods have large computational order of convergence (COC) than all the existing methods which sign that our newly proposed iterative methods are well-matched to inspect the roots.

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طرائق تكرارية كفوءة من ثلاث خطوات تعتمد على صيغة برنشتاين التربيعية لحل المعادلات غير الخطية

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المستخلص

في هذه الدراسة، نقترح ونحلل عائلتين جديدتين من الطرائق التكرارية الكفؤة مع معلمة واحدة خالية من المشتقات العليا لحل المعادلات غير الخطية بالاعتماد على نظرية نيوتن في حساب التفاضل والتكامل وصيغة برنشتاين التربيعية وأساسات متعددة حدود برنولي ونشر تايلور وبعض التقنيات العددية. نثبت أن الطرائق التكرارية الجديدة تصل الى رتبة تقارب عشرة مع ستة دوال لكل تكرار وثمانية مع أربع دوال لكل تكرار مما يعني أن مؤشر كفاءة الطرائق التكرارية الجديدة هو 1.4678 و1.6818 على التوالي. تقديم أمثلة عددية لإظهار كفاءة وأداء طرائقنا التكرارية، مقارنة بطريقة نيوتن وغيرها من الطرائق ذات الصلة.