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A Penalized Likelihood Estimator for Variance Components in Repeated Measurements Model

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Abstract

In this paper, we consider the repeated measurement model. We estimate the parameters of the model by using the maximum likelihood method and compared these estimators with the penalized likelihood estimators by using mean square error. We also determine the corresponding moments of each estimators. Finally, we find bias equations and variance for each estimator.

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1. Introduction

Repeated measurements analysis is widely used in many fields, for example, the health and life sciences, epidemiology, biomedical, agricultural, industrial, psychological, and education and so on [1, 2]. Repeated measurements is a term used to describe data in which the response variable for each experimental unit is observed on multiple occasions and possibly under different experimental conditions [2]. Repeated measures designs involving two or more independent groups are among the most common experimental designs in a variety of research settings. Various statistical procedures have been suggested for analyzing data from split-plot designs when parametric model assumptions are violated [3]. Maximum Likelihood is a useful way to estimate the variance parameters[4]. Penalized Likelihood estimation has been used to obtain more stable estimates in item response theory [4]. Al-Mouel and Mustafa [5] studied the lawley-Hotelling test in one-way multivariate repeated measurements model. Al-Mouel and Swadi [6] studied the Bayesian estimators of the one-way repeated measurements model. Al-Mouel and Al-Shmailawi [7] studied the modified maximum likelihood estimators for one-way repeated measurements model. Al-Mouel and Al-Shmailawi [8] studied the sufficiency and completeness of the one-way repeated measurements model. Al-Mouel and Al-Isawi [9] computed the quadratic unbiased estimator, which has minimum variance (best quadratic unbiased estimate) by using analysis of variance (ANOVA) method of estimating the variance components. Kursawe and et al [10] revealed the importance of repeated measurements for parameterizing cell-based models of growing tissues. Al-Isawi [11] proposed a new approach to determine the matrices for minimum variance quadratic unbiased estimators of variance components and minimum norm quadratic unbiased estimators of variance components for the repeated measurements model. Mcfarquhar [12] studied the group level repeated measurements of neuroimaging data using the univariate general linear model. Al-Mouel and Afnan [13] studied the Bayesian approach on one-way repeated measurements model. The objective of this paper is to estimate the variance components of the model by using maximum likelihood method, we use the mean square error to compare the maximum likelihood estimator with the penalized likelihood estimators. We also determine the corresponding moments for each estimator. We find bias equations and variance for each estimator.

2. Setting up the Model

We considered the repeated measurement model as follows

$$y_{ijk} = \mu + \tau_j + \gamma_k + \delta_{i(j)} + \lambda_{i(k)} + (\tau\gamma)_{jk} + e_{ijk} \quad (1)$$

$i = 1, \dots, n$ is an index for experimental unit,

$j = 1, \dots, q$ is an index for levels of between unit factor,

$k = 1, \dots, p$ is an index for levels of within-unit factor,

y_{ijk} is the response measurement at within unit factor k for unit i within between-unit factor j,

τ_j is the added effect for treatment between-unit, factor j,

γ_k is the added effect for within-units factor k,

$\delta_{i(j)}$ is the random effect due to the experimental unit i within treatment between-units factor j,

$\lambda_{i(k)}$ is the random effect due to the experimental unit i within-units factor k,

$(\tau\gamma)_{jk}$ is the added effect for between-unit, factor $j \times$ within-unit factor k interaction, and

e_{ijk} is the random error on within-unit factor k for unit i within between-unit factor j .

For the parametrization to be of full rank we impose the following set of conditions :

$$\sum_{j=1}^{q} \tau_j = 0, \sum_{j=1}^{q} (\tau\gamma)_{jk} = 0, \; for\ each\ k = 1, \dots, \text{p} \quad (2)$$

$$\sum_{k=1}^{\text{p}} \gamma_k = 0, \sum_{k=1}^{\text{p}} (\tau\gamma)_{jk} = 0, \; for\ each\ j = 1, \dots, \text{q} \quad (3)$$

e_{ijk} , $\delta_{i(j)} and\ \lambda_{i(k)}$ are independent with

$e_{ijk} \sim i.i.d\ N(0, \sigma_e^2)$, $\delta_{i(j)} \sim i.i.d\ N\left(0, \sigma_\delta^2\right)$, $\lambda_{i(k)} \sim i.i.d\ N\left(0, \sigma_\lambda^2\right)$

The expressions of $SS_G, SS_W, SS_{U(G)}, SS_{G\times W}, and\ SS_E$ are

$$SS_G = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} \left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2, \quad (4)$$

$$SS_W = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} (\bar{y}_{..k} - \bar{y}_{...})^2, \quad (5)$$

$$SS_{U(G)} = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} \left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2 + \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} (\bar{y}_{i.k} - \bar{y}_{..k})^2, \quad (6)$$

$$SS_{G\times W} = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} \left(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}\right)^2, \quad (7)$$

and

$$SS_E = \sum_{i=1}^{n} \sum_{j=1}^{q} \sum_{k=1}^{\text{P}} \left(\left(y_{ijk} - \bar{y}_{ij.} - \bar{y}_{i.k} - \bar{y}_{.jk} + \bar{y}_{i..} + \bar{y}_{.j.} + \bar{y}_{..k} - \bar{y}_{...}\right)\right)^2. \quad (8)$$

$\boldsymbol{Lemma}(\mathbf{1})[\mathbf{14}]$. Let $z_1, z_2, \dots, z_n$ be independent normal variables with mean μ and variance σ^2. Then $\sum_{i=1}^{n} (z_i - \bar{z})^2 \sim \sigma^2 \chi^2_{[n-1]}$.

3. Maximum likelihood estimators

To estimate the parameters $\sigma_\delta^2, \sigma_\lambda^2, and\ \sigma_e^2$, we use the maximum likelihood method which aim to make the likelihood function at the maximum values.

Theorm(1). The maximum likelihood estimators of variance components are

$$\hat{\sigma}_e^2 = \frac{SS_E + SS_{G\times W}}{v_e + v_{g\times w}},\ \hat{\sigma}_\delta^2 = \frac{1}{p}\left(\frac{SS_G}{v_g} - \frac{SS_E + SS_{G\times W}}{v_e + v_{g\times w}}\right), \text{ and } \hat{\sigma}_\lambda^2 = \frac{1}{q}\left(\frac{SS_W}{v_w} - \frac{SS_E + SS_{G\times W}}{v_e + v_{g\times w}}\right). \quad (9)$$

Proof. The maximum likelihood estimators of $\sigma_\delta^2, \sigma_\lambda^2, and\ \sigma_e^2$, can be developed by starting from the distribution of the sufficient statistics $\bar{y}_{\dots}, SS_E, SS_G, SS_W, SS_{G\times W}, and\ SS_{U(G)}$. By applying the factorization theorem of sufficient statistics and lemma(1), then the quantities $\bar{y}_{\dots}, SS_E, SS_G, SS_W, SS_{G\times W}, and\ SS_{U(G)}$ are constitute a sufficient statistics of $\mu, \sigma_e^2, \sigma_\delta^2, and\ \sigma_\lambda^2$. The likelihood function is given by

$$L = \frac{exp\left[\frac{-1}{2}\left\{\frac{SS_E}{\sigma_e^2} + \frac{SS_G}{p\sigma_\delta^2 + \sigma_e^2} + \frac{SS_W}{q\sigma_\lambda^2 + \sigma_e^2} + \frac{SS_{G\times W}}{\sigma_e^2} + \frac{(q+p-1)SS_{U(G)}}{qp\left(\sigma_\delta^2 + \sigma_\lambda^2\right)} + \frac{nqp(\bar{y}_{\dots} - \mu)^2}{p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2}\right\}\right]}{(2\pi)^{\frac{1}{2}nqp}(\sigma_e^2)^{\frac{1}{2}v_e}(p\sigma_\delta^2 + \sigma_e^2)^{\frac{1}{2}v_g}(q\sigma_\lambda^2 + \sigma_e^2)^{\frac{1}{2}v_w}(\sigma_e^2)^{\frac{1}{2}v_{g\times w}}\left(\frac{qp\left(\sigma_\delta^2 + \sigma_\lambda^2\right)}{q+p-1}\right)^{\frac{1}{2}v_{u(g)}}(p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2)^{\frac{1}{2}}}, \quad (10)$$

Where $v_e = (n-1)(q-1)(p-1), v_g = (q-1), v_w = (p-1), v_{g\times w} = (q-1)(p-1)$, and $v_{u(g)} = (n-1)(q+p-1)$.

The log-likelihood function is given by

$$ln(L) = \frac{-1}{2}\Bigg[nqp\ ln(2\pi) + v_e\ ln(\sigma_e^2) + v_g\ ln\left(p\sigma_\delta^2 + \sigma_e^2\right) + v_w\ ln\left(q\sigma_\lambda^2 + \sigma_e^2\right) + v_{g\times w}\ ln(\sigma_e^2)$$
$$+ v_{u(g)}\ ln\left(\sigma_\delta^2 + \sigma_\lambda^2\right) + ln\left(p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2\right) + \frac{SS_E}{\sigma_e^2} + \frac{SS_G}{p\sigma_\delta^2 + \sigma_e^2} + \frac{SS_W}{q\sigma_\lambda^2 + \sigma_e^2}$$
$$+ \frac{SS_{G\times W}}{\sigma_e^2} + \frac{(q+p-1)SS_{U(G)}}{qp\left(\sigma_\delta^2 + \sigma_\lambda^2\right)}$$
$$+ \frac{nqp(\bar{y}_{\dots} - \mu)^2}{p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2}\Bigg]. \quad (11)$$

Equating to zero the partial derivatives of $ln(L)$ with respect to σ_e^2, $\left(p\sigma_\delta^2 + \sigma_e^2\right)$, and $\left(q\sigma_\lambda^2 + \sigma_e^2\right)$, we obtain

$$\frac{\partial ln(L)}{\partial \sigma_e^2} = \frac{v_e}{\sigma_e^2} - \frac{SS_E}{(\sigma_e^2)^2} + \frac{v_{g\times w}}{\sigma_e^2} - \frac{SS_{G\times W}}{(\sigma_e^2)^2} = 0 \quad (12)$$

$$\frac{\partial ln(L)}{\partial (p\sigma_\delta^2+\sigma_e^2)} = \frac{v_g}{p\sigma_\delta^2+\sigma_e^2} - \frac{SS_G}{(p\sigma_\delta^2+\sigma_e^2)^2} = 0 \quad (13)$$

$$\frac{\partial ln(L)}{\partial (q\sigma_\lambda^2+\sigma_e^2)} = \frac{v_w}{(q\sigma_\lambda^2+\sigma_e^2)} - \frac{SS_W}{(q\sigma_\lambda^2+\sigma_e^2)^2} = 0 \quad (14)$$

it follows that $\hat{\sigma}_e^2 = \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}}$, $\hat{\sigma}_\delta^2 = \frac{1}{p}\left(\frac{SS_G}{v_g} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}}\right)$, (15)

and $\hat{\sigma}_\lambda^2 = \frac{1}{q}\left(\frac{SS_W}{v_w} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}}\right)$. (16)

Theorem(2) The maximum likelihood estimators have the following bias equations, respectively,

a- $E(\hat{\sigma}_e^2) - \sigma_e^2 = 0,$ (17)

b- $E(\hat{\sigma}_\delta^2) - \sigma_\delta^2 = 0,$ (18)

c- $E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = 0.$ (19)

Proof. From lemma (1), we have

$$SS_E \sim \sigma_e^2 \chi_{v_e}^2, \quad (20)$$

$$\frac{SS_G}{(\sigma_e^2 + p\sigma_\delta^2)} \sim \chi_{(v_g)}^2, where\ \tau_j = 0, \quad (21)$$

$$\frac{SS_W}{(\sigma_e^2 + q\sigma_\lambda^2)} \sim \chi_{(v_w)}^2, where\ \gamma_k = 0, \quad (22)$$

$$\frac{SS_{G\times W}}{\sigma_e^2} \sim \chi_{(v_{g\times w})}^2, where\ (\tau\gamma)_{jk} = 0, \quad (23)$$

a- $E(\hat{\sigma}_e^2) - \sigma_e^2 = E\left(\frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}}\right) - \sigma_e^2 = \frac{1}{v_e+v_{g\times w}}\left(E(SS_E) + E(SS_{G\times W})\right) - \sigma_e^2 = \frac{1}{v_e+v_{g\times w}}(v_e + v_{g\times w})\sigma_e^2 - \sigma_e^2 = 0.$ (24)

b- $E(\hat{\sigma}_\delta^2) - \sigma_\delta^2 = E\left(\frac{1}{p}\left(\frac{SS_G}{v_G} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}}\right)\right) - \sigma_\delta^2 = \frac{1}{p}\left(E\left(\frac{SS_G}{v_G}\right) - E(\hat{\sigma}_e^2)\right) - \sigma_\delta^2 = \frac{1}{p}\left(\frac{1}{v_g}(p\sigma_\delta^2 + \sigma_e^2)v_g - \sigma_e^2\right) - \sigma_\delta^2 = 0.$ (25)

c- $E\left(\hat{\sigma}_{\lambda}^{2}\right)-\sigma_{\lambda}^{2}=E\left(\frac{1}{q}\left(\frac{SS_{W}}{w}-\frac{SS_{E}+SS_{G\times W}}{v_{e}+v_{g\times w}}\right)\right)-\sigma_{\lambda}^{2}=\frac{1}{q}\left(E\left(\frac{SS_{W}}{w}\right)-E\left(\hat{\sigma}_{e}^{2}\right)\right)-\sigma_{\lambda}^{2}=$

$$\frac{1}{q}\left(\frac{1}{v_{w}}\left(q\sigma_{\lambda}^{2}+\sigma_{e}^{2}\right)v_{w}-\sigma_{e}^{2}\right)-\sigma_{\lambda}^{2}=0. \qquad (26)$$

Theorem(3) The maximum likelihood estimators have the following variance, respectively,

a- $var\left(\hat{\sigma}_{e}^{2}\right)=\frac{2\left(\sigma_{e}^{2}\right)^{2}}{\left(v_{e}+v_{g\times w}\right)},$ (27)

b- $var\left(\hat{\sigma}_{\delta}^{2}\right)=\frac{2\left(p\,\sigma_{\delta}^{2}+\sigma_{e}^{2}\right)^{2}}{p^{2}\,v_{g}}+\frac{2\left(\sigma_{e}^{2}\right)^{2}}{p^{2}\left(v_{e}+v_{g\times w}\right)},$ (28)

c- $var\left(\hat{\sigma}_{\lambda}^{2}\right)=\frac{2\left(q\,\sigma_{\lambda}^{2}+\sigma_{e}^{2}\right)^{2}}{q^{2}\,v_{w}}+\frac{2\left(\sigma_{e}^{2}\right)^{2}}{q^{2}\left(v_{e}+v_{g\times w}\right)}.$ (29)

Proof. From lemma 1 (Equs. 20, 21, 22, and 23)

a- $var\left(\hat{\sigma}_{e}^{2}\right)=var\left(\frac{SS_{E}+SS_{G\times W}}{v_{e}+v_{g\times w}}\right)=\frac{1}{\left(v_{e}+v_{g\times w}\right)^{2}}\left(var\left(SS_{E}\right)+var\left(SS_{G\times W}\right)\right)=$

$$\frac{1}{\left(v_{e}+v_{g\times w}\right)^{2}}\left(2\,v_{e}\left(\sigma_{e}^{2}\right)^{2}+2\,v_{g\times w}\left(\sigma_{e}^{2}\right)^{2}\right)=\frac{2\left(\sigma_{e}^{2}\right)^{2}}{\left(v_{e}+v_{g\times w}\right)}. \qquad (30)$$

b- $var\left(\hat{\sigma}_{\delta}^{2}\right)=\frac{1}{p^{2}}\left(var\left(\frac{SS_{G}}{v_{g}}\right)+var\left(\frac{SS_{E}+SS_{G\times W}}{v_{e}+v_{g\times w}}\right)\right)=\frac{1}{p^{2}}\left(\frac{2\left(p\,\sigma_{\delta}^{2}+\sigma_{e}^{2}\right)^{2}}{v_{g}}+var\left(\hat{\sigma}_{e}^{2}\right)\right)=$

$$\frac{2\left(p\,\sigma_{\delta}^{2}+\sigma_{e}^{2}\right)^{2}}{p^{2}\,v_{g}}+\frac{2\left(\sigma_{e}^{2}\right)^{2}}{p^{2}\left(v_{e}+v_{g\times w}\right)}. \qquad (31)$$

c- $var\left(\hat{\sigma}_{\lambda}^{2}\right)=\frac{1}{q^{2}}\left(var\left(\frac{SS_{W}}{v_{w}}\right)+var\left(\frac{SS_{E}+SS_{G\times W}}{v_{e}+v_{g\times w}}\right)\right)=\frac{1}{q^{2}}\left(\frac{2\left(q\,\sigma_{\lambda}^{2}+\sigma_{e}^{2}\right)^{2}}{v_{w}}+var\left(\hat{\sigma}_{e}^{2}\right)\right)=$

$$\frac{2\left(q\,\sigma_{\lambda}^{2}+\sigma_{e}^{2}\right)^{2}}{q^{2}\,v_{w}}+\frac{2\left(\sigma_{e}^{2}\right)^{2}}{q^{2}\left(v_{e}+v_{g\times w}\right)}. \qquad (32)$$

4. Maximum Penalized likelihood estimators

In this section, we specify a penalty for $\sigma_{e}^{2},\sigma_{\delta}^{2}$, and σ_{λ}^{2}, the penalized log-likelihood function is given by

$$ln\left(L_{p}\right)=ln(L)+\ln p\left(\sigma_{e}^{2},\sigma_{\delta}^{2},\sigma_{\lambda}^{2}\right), \qquad (33)$$

Where the first term of the right-hand side is the log-likelihood and $\ln p\left(\sigma_{e}^{2},\sigma_{\delta}^{2},\sigma_{\lambda}^{2}\right)$ is an additive penalty term. We find the maximum penalized likelihood estimators by maximizing

(33). The exponential of the penalty term can be regard as a Bayesian prior density for $\sigma_e^2, \sigma_\delta^2$, and σ_λ^2.

Theorm(4). The maximum penalized likelihood estimators of variance components are

$$\hat{\sigma}_e^2 = \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2},\ \hat{\sigma}_\delta^2 = \frac{1}{p}\left(\frac{SS_G}{v_g+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right), \text{ and } \hat{\sigma}_\lambda^2 = \frac{1}{q}\left(\frac{SS_W}{v_w+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right). \quad (34)$$

Proof. The penalized log-likelihood function is

$$ln(L_p) = \frac{-1}{2}\Bigg[nqp\ ln(2\pi) + v_e\ ln(\sigma_e^2) + v_g\ ln(p\sigma_\delta^2 + \sigma_e^2) + v_w\ ln(q\sigma_\lambda^2 + \sigma_e^2) + v_{g\times w}\ ln(\sigma_e^2) + v_{u(g)}\ ln(\sigma_\delta^2 + \sigma_\lambda^2) + ln(p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2) + \frac{SS_E}{\sigma_e^2} + \frac{SS_G}{p\sigma_\delta^2 + \sigma_e^2} + \frac{SS_W}{q\sigma_\lambda^2 + \sigma_e^2} + \frac{SS_{G\times W}}{\sigma_e^2} + \frac{(q+p-1)SS_{U(G)}}{qp(\sigma_\delta^2 + \sigma_\lambda^2)} + \frac{nqp(\bar{y}_{\ldots} - \mu)^2}{p\sigma_\delta^2 + q\sigma_\lambda^2 + \sigma_e^2}\Bigg] - ln(\sigma_e^2) - ln(p\sigma_\delta^2 + \sigma_e^2) - ln(q\sigma_\lambda^2 + \sigma_e^2). \quad (35)$$

Where the additive penalty is defined as the log-likelihood for the Bayesian prior density. we consider the prior distribution for $\mu, \sigma_e^2, \sigma_\delta^2, and\ \sigma_\lambda^2$ as follows

$$p(\mu, \sigma_e^2, \sigma_\delta^2, \sigma_\lambda^2) \propto \frac{1}{\sigma_e^2(p\sigma_\delta^2 + \sigma_e^2)(q\sigma_\lambda^2 + \sigma_e^2)}. \quad (36)$$

Equating to zero the partial derivatives of $ln(L_p)$ with respect to $\sigma_e^2, (p\sigma_\delta^2 + \sigma_e^2)$, and $(q\sigma_\lambda^2 + \sigma_e^2)$ we obtain

$$\frac{\partial ln(L_p)}{\partial \sigma_e^2} = \frac{v_e}{\sigma_e^2} - \frac{SS_E}{(\sigma_e^2)^2} + \frac{v_{g\times w}}{\sigma_e^2} - \frac{SS_{G\times W}}{(\sigma_e^2)^2} + \frac{2}{\sigma_e^2} = 0 \quad (37)$$

$$\frac{\partial ln(L_p)}{\partial (p\sigma_\delta^2 + \sigma_e^2)} = \frac{v_g}{p\sigma_\delta^2 + \sigma_e^2} - \frac{SS_G}{(p\sigma_\delta^2 + \sigma_e^2)^2} + \frac{2}{p\sigma_\delta^2 + \sigma_e^2} = 0 \quad (38)$$

$$\frac{\partial ln(L_p)}{\partial (q\sigma_\lambda^2 + \sigma_e^2)} = \frac{v_w}{(q\sigma_\lambda^2 + \sigma_e^2)} - \frac{SS_W}{(q\sigma_\lambda^2 + \sigma_e^2)^2} + \frac{2}{(q\sigma_\lambda^2 + \sigma_e^2)} = 0 \quad (39)$$

it follows that $\hat{\sigma}_e^2 = \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2},\ \hat{\sigma}_\delta^2 = \frac{1}{p}\left(\frac{SS_G}{v_g+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right),$ (40)

and $\hat{\sigma}_\lambda^2 = \frac{1}{q}\left(\frac{SS_W}{v_w+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right)$. (41)

Theorem(5) The maximum penalized likelihood estimators have the following bias equations, respectively,

a- $E(\hat{\sigma}_e^2) - \sigma_e^2 = \frac{(v_e+v_{g\times w})\sigma_e^2}{(v_e+v_{g\times w}+2)} - \sigma_e^2,$ (42)

b- $E(\hat{\sigma}_\delta^2) - \sigma_\delta^2 = \frac{v_g(p\sigma_\delta^2+\sigma_e^2)}{p\,(v_g+2)} - \frac{(v_e+v_{g\times w})\sigma_e^2}{p\,(v_e+v_{g\times w}+2)} - \sigma_\delta^2,$ (43)

c- $E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = \frac{v_w(q\sigma_\lambda^2+\sigma_e^2)}{q\,(v_w+2)} - \frac{(v_e+v_{g\times w})\sigma_e^2}{q\,(v_e+v_{g\times w}+2)} - \sigma_\lambda^2.$ (44)

Proof. From lemma1(Equs. 20, 21, 22, and 23)

a- $E(\hat{\sigma}_e^2) - \sigma_e^2 = E\left(\frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right) - \sigma_e^2 = \frac{1}{v_e+v_{g\times w}+2}\left(E(SS_E) + E(SS_{G\times W})\right) - \sigma_e^2 =$
$\frac{1}{v_e+v_{g\times w}+2}(v_e + v_{g\times w})\sigma_e^2 - \sigma_e^2.$ (45)

b- $E(\hat{\sigma}_\delta^2) - \sigma_\delta^2 = E\left(\frac{1}{p}\left(\frac{SS_G}{v_g+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right)\right) - \sigma_\delta^2 = \frac{1}{p}\left(E\left(\frac{SS_G}{v_g+2}\right) - E(\hat{\sigma}_e^2)\right) - \sigma_\delta^2 =$
$\frac{1}{p}\left(\frac{1}{v_g+2}(p\sigma_\delta^2 + \sigma_e^2)v_g - \frac{1}{v_e+v_{g\times w}+2}(v_e + v_{g\times w})\sigma_e^2\right) - \sigma_\delta^2 = \frac{v_g(p\sigma_\delta^2+\sigma_e^2)}{p\,(v_g+2)} - \frac{(v_e+v_{g\times w})\sigma_e^2}{p\,(v_e+v_{g\times w}+2)} -$
$\sigma_\delta^2.$ (46)

c- $E(\hat{\sigma}_\lambda^2) - \sigma_\lambda^2 = E\left(\frac{1}{q}\left(\frac{SS_W}{v_w+2} - \frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w+2}}\right)\right) - \sigma_\lambda^2 = \frac{1}{q}\left(E\left(\frac{SS_W}{v_{w+2}}\right) - E(\hat{\sigma}_e^2)\right) - \sigma_\lambda^2 =$
$\frac{1}{q}\left(\frac{1}{v_w+2}(q\sigma_\lambda^2 + \sigma_e^2)v_w - \frac{1}{v_e+v_{g\times w}+2}(v_e + v_{g\times w})\sigma_e^2\right) - \sigma_\lambda^2 = \frac{v_w(q\sigma_\lambda^2+\sigma_e^2)}{q\,(v_w+2)} - \frac{(v_e+v_{g\times w})\sigma_e^2}{q\,(v_e+v_{g\times w}+2)} -$
$\sigma_\lambda^2.$ (47)

Theorem(6) The maximum penalized likelihood estimators have the following variance, respectively,

a- $var(\hat{\sigma}_e^2) = \frac{2\,(v_e+v_{g\times w})(\sigma_e^2)^2}{(v_e+v_{g\times w}+2)^2},$ (48)

b- $var(\hat{\sigma}_\delta^2) = \frac{2v_g(v_g+2)^2(p\,\sigma_\delta^2+\sigma_e^2)^2}{p^2} + \frac{2(v_e+v_{g\times w})\,(\sigma_e^2)^2}{p^2(v_e+v_{g\times w}+2)^2},$ (49)

c- $var(\hat{\sigma}_\lambda^2) = \frac{2v_w(v_w+2)^2(q\,\sigma_\lambda^2+\sigma_e^2)^2}{q^2} + \frac{2(v_e+v_{g\times w})\,(\sigma_e^2)^2}{q^2(v_e+v_{g\times w}+2)^2}.$ (50)

Proof. From lemma 1 (Equs. 20, 21, 22, and 23)

a- $var(\hat{\sigma}_e^2) = var\left(\frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right) = \frac{1}{(v_e+v_{g\times w}+2)^2}\left(var(SS_E)+var(SS_{G\times W})\right) =$

$\frac{1}{(v_e+v_{g\times w}+2)^2}\left(2\,v_e(\sigma_e^2)^2 + 2\,v_{g\times w}(\sigma_e^2)^2\right) = \frac{2(v_e+v_{g\times w})\,(\sigma_e^2)^2}{(v_e+v_{g\times w}+2)^2}.$ (51)

b- $var(\hat{\sigma}_\delta^2) = \frac{1}{p^2}\left(var\left(\frac{SS_G}{v_g+2}\right) + var\left(\frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right)\right) = \frac{1}{p^2}\left(2v_g(v_g+2)^2(p\,\sigma_\delta^2+\sigma_e^2)^2 +\right.$

$\left. var(\hat{\sigma}_e^2)\right) = \frac{2v_g(v_g+2)^2(p\,\sigma_\delta^2+\sigma_e^2)^2}{p^2} + \frac{2(v_e+v_{g\times w})\,(\sigma_e^2)^2}{p^2(v_e+v_{g\times w}+2)^2}.$ (52)

c- $var(\hat{\sigma}_\lambda^2) = \frac{1}{q^2}\left(var\left(\frac{SS_W}{v_w+2}\right) + var\left(\frac{SS_E+SS_{G\times W}}{v_e+v_{g\times w}+2}\right)\right) = \frac{1}{q^2}\left(2v_w(v_w+2)^2(q\,\sigma_\lambda^2+\sigma_e^2)^2 +\right.$

$\left. var(\hat{\sigma}_e^2)\right) = \frac{2v_w(v_w+2)^2(q\,\sigma_\lambda^2+\sigma_e^2)^2}{q^2} + \frac{2(v_e+v_{g\times w})\,(\sigma_e^2)^2}{q^2(v_e+v_{g\times w}+2)^2}.$ (53)

5. Difference between maximum likelihood method and maximum penalized likelihood method

The following theorem used to show the relationship between the estimators of variance components and compare these estimators by using mean square error.

***Theorem*(7).** Uniformly in the parameters

a- $MSE(\hat{\sigma}_{e,ML}^2) \le MSE(\hat{\sigma}_{e,MPL}^2),$ (54)

b- $MSE(\hat{\sigma}_{\delta,ML}^2) \le MSE(\hat{\sigma}_{\delta,MPL}^2),$ (55)

c- $MSE(\hat{\sigma}_{\lambda,ML}^2) \le MSE(\hat{\sigma}_{\lambda,MPL}^2).$ (56)

Proof.

a- $MSE\left(\hat{\sigma}^2_{e,ML}\right) = \frac{2\left(\sigma_e^2\right)^2}{\left(v_e+v_{g\times w}\right)} \leq MSE\left(\hat{\sigma}^2_{e,MPL}\right) = \frac{2\left(v_e+v_{g\times w}\right)\left(\sigma_e^2\right)^2}{\left(v_e+v_{g\times w}+2\right)^2} + \left(\frac{\left(v_e+v_{g\times w}\right)\sigma_e^2}{\left(v_e+v_{g\times w}+2\right)} - \sigma_e^2\right)^2,$

b- $MSE\left(\hat{\sigma}^2_{\delta,ML}\right) = \frac{2\left(p\sigma_\delta^2+\sigma_e^2\right)^2}{p^2 v_g} + \frac{2\left(\sigma_e^2\right)^2}{p^2\left(v_e+v_{g\times w}\right)} \leq MSE\left(\hat{\sigma}^2_{\delta,MPL}\right) = \frac{2v_g\left(v_g+2\right)^2\left(p\,\sigma_\delta^2+\sigma_e^2\right)^2}{p^2} +$
$\frac{2\left(v_e+v_{g\times w}\right)\left(\sigma_e^2\right)^2}{p^2\left(v_e+v_{g\times w}+2\right)^2} + \left(\frac{v_g\left(p\sigma_\delta^2+\sigma_e^2\right)}{p\left(v_g+2\right)} - \frac{\left(v_e+v_{g\times w}\right)\sigma_e^2}{p\left(v_e+v_{g\times w}+2\right)} - \sigma_\delta^2\right)^2,$

c- $MSE\left(\hat{\sigma}^2_{\lambda,ML}\right) = \frac{2\left(q\sigma_\lambda^2+\sigma_e^2\right)^2}{q^2 v_w} + \frac{2\sigma_e^2}{q^2\left(v_e+v_{g\times w}\right)} \leq MSE\left(\hat{\sigma}^2_{\lambda,MPL}\right) = \frac{2v_w(v_w+2)^2\left(q\,\sigma_\lambda^2+\sigma_e^2\right)^2}{q^2} +$
$\frac{2\left(v_e+v_{g\times w}\right)\left(\sigma_e^2\right)^2}{q^2\left(v_e+v_{g\times w}+2\right)^2} + \left(\frac{v_w\left(q\sigma_\lambda^2+\sigma_e^2\right)}{q\left(v_w+2\right)} - \frac{\left(v_e+v_{g\times w}\right)\sigma_e^2}{q\left(v_e+v_{g\times w}+2\right)} - \sigma_\lambda^2\right)^2. \quad (57)$

Conclusions

For the repeated measurements model, we conclude the following:

1- The maximum likelihood estimators are unbiased.
2- The mean square error is used to show the relationship between the estimators of variance components, such that

a- $MSE\left(\hat{\sigma}^2_{e,ML}\right) \leq MSE\left(\hat{\sigma}^2_{e,MPL}\right)$,
b- $MSE\left(\hat{\sigma}^2_{\delta,ML}\right) \leq MSE\left(\hat{\sigma}^2_{\delta,MPL}\right)$,
c- $MSE\left(\hat{\sigma}^2_{\lambda,ML}\right) \leq MSE\left(\hat{\sigma}^2_{\lambda,MPL}\right)$.

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مقدر الاحتمالية المعاقب لمركبات التباين في نموذج القياسات المتكررة

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المستخلص

في هذا البحث، قمنا باعتماد نموذج القياسات المتكررة، قمنا بتقدير معلمات النموذج باستخدام طريقة الاحتمال القصوى، و قمنا بمقارنة هذه المقدرات مع مقدرات الاحتمال المعاقبة باستخدام متوسط الخطأ المربع. كما حددنا العزوم المقابلة لكل مقدر. و اخيرا، وجدنا معادلات التحيز و التباين لكل مقدر.