Double Sums of the Szasz Sequence

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ARTICLE INFO	**ABSTRACT**
Keywords Ordinary approximation, Voronovskaya-type, asymptotic formula, linear positive operators, Szasz sequence.	This study introduced and investigated a modification of the Szasz sequence in double sums with one variable. This sequence's recurrence relation for the m-th order moment, convergence theorem, and Voronovskaya-type asymptotic theorem in ordinary approximation are investigated. A numerical example is also provided to demonstrate the approximation of the chosen test function by this modification sequence, and the numerical results are compared to the numerical results of the classical Szasz sequence. It turns out that the results achieved from the modification sequence are superior to those obtained from the classical sequence.

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1. Introduction

In 1950, the well-known classical sequence of Szasz is defined as [1]:

$$M_n(f;x) \coloneqq \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1}$$

where, $q_{n,k}(x) := \frac{(nx)^k}{e^{nx}\, k!}$.

In 1969, Jakimovski and Leviatan modified the Sazsz sequence using the Apple polynomials called the Szasz-Jakimovski-Leviatan sequence [2].In 1974, the Szasz-Jakimovski-Leviatan sequence is generalized by using Sheffer polynomials [3]. In 1988, to establish a summation-integral type on the space of integrable functions on $[0,\infty)$ suggest a series of modified Sazsz operators [4]. In 1994-1997, Ciup provides and establishes a few approximation properties for the operators (1); for additional information, see [5-6]. In 2020, the Szasz sequence is generalized using a multiple of Sheffer polynomials [7]. For more details see [8-11]. For a function $h \in C_\alpha[0,\infty) \coloneqq \{\, h \in C[0,\infty); |h(t)| \leq Me^{\alpha t} \text{ for some } \alpha > 0\}$ and $s \geq -\frac{1}{2}$, the sequence $G_{n,s}(h;x)$ is defined as:

$$G_{n,s}(h;x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} h\left(x + \frac{1}{n^s}(\frac{k+j}{n} - x)\right) S_{j,k}(x)\ , \tag{2}$$

where, $S_{j,k}(x) = \frac{\left(\frac{nx}{2}\right)^{k+j}}{e^{nx} k!\, j!}$.

2. Preliminary Results

some lemmas that are used in the main results are introduced and proven. The first lemma shows that the relationship between $S_{j,k}(x)$ and $S'_{j,k}(x)$.

Lemma 2.1.

For $n \in N$ the function $S_{j,k}(x) = \frac{\left(\frac{nx}{2}\right)^{k+j}}{e^{nx} k!\, j!}$ possesses the following property:

$$\frac{x}{n}S'_{j,k}(x) = \left(\frac{k+j}{n} - x\right)S_{j,k}(x). \tag{3}$$

Proof.

Using the derivative for $S_{r_i}(x)$ for x,

$$S'_{j,k}(x) = \frac{n}{2}(k+j)\frac{\left(\frac{nx}{2}\right)^{k+j-1}}{k!\,j!}e^{-nx} - ne^{-nx}\frac{\left(\frac{nx}{2}\right)^{k+j}}{k!\,j!}$$

$$\frac{x}{n}S'_{j,k}(x) = \left(\frac{k+j}{n} - x\right)S_{j,k}(x). \quad \blacksquare$$

For $m \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, the moment of the sequence $G_{n,s}(h;x)$ from the order, m-th is defined as:

$$T_{n,m}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S_{j,k}(x)\left(\frac{1}{n^s}\left(\frac{k+j}{n} - x\right)\right)^m. \tag{4}$$

Lemma 2.2.

$\forall m \in \mathbb{N}^0$, The recurrence relation for the function $T_{n,m}(x)$ is as follows:

$$T_{n,m+1}(x) = \frac{x}{n^{s+1}}\left(\frac{m}{n^s}T_{n,m-1}(x) + T'_{n,m}(x)\right). \tag{5}$$

Proof.

Derive both sides of the equation (4) about x, one gets

$$T'_{n,m}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S'_{j,k}(x)\left(\frac{1}{n^s}\left(\frac{k+j}{n} - x\right)\right)^m - \frac{m}{n^{sm}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S'_{j,k}(x)\left(\frac{k+j}{n} - x\right)^{m-1}$$

from Lemma 2.1, one has

$$T_{n,m+1}(x) = \frac{x}{n^{s+1}}\left(\frac{m}{n^s}T_{n,m-1}(x) + T'_{n,m}(x)\right). \blacksquare$$

The next lemma contains some properties of the sequence $G_{n,s}(.\,;x)$.

Lemma 2.3.

Suppose that $h(t) = t^m, m \in \mathbb{N}^0$, then the following properties are held:

(i) $G_{n,s}(1; x) = 1;$

(ii) $G_{n,s}(t; x) = x;$

(iii) $G_{n,s}(t^2; x) = x^2 + \frac{x}{n^{2s+1}};$

(iv) $G_{n,s}(t^m; x) = C_1 x^m + C_2 x^{m-1} + T.L.P.(x), \forall m \in \mathbb{N}^0,$

where,

$$C_1 \coloneqq \sum_{i=0}^{m} \binom{m}{i} \sum_{\ell=0}^{i} \frac{(n^s - 1)^{m-i}}{n^{sm}} \binom{i}{\ell}, \tag{6}$$

and

$$C_2 := \sum_{i=0}^{m} \sum_{\ell=0}^{i} \left(\frac{(i-\ell)(i-\ell-1) + \ell(\ell-1)}{2} \right) \frac{(n^s - 1)^{m-i}}{n^{sm+1}} \binom{i}{\ell}. \tag{7}$$

Proof.

The consequence (i) is held immediately from (2), and (ii) is proved as follows,

$$G_{n,s}(t; x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(x + \frac{1}{n^s} \left(\frac{k+j}{n} - x \right) \right) S_{j,k}(x)$$

$$= x \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S_{j,k}(x) + \frac{1}{n^s} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{k+j}{n} - x \right) S_{j,k}(x)$$

$$= x.$$

In (iii) using the same way (ii), one has

$$G_{n,s}(t^2; x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(x + \frac{1}{n^s} \left(\frac{k+j}{n} - x \right) \right)^2 S_{j,k}(x)$$

$$= x^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x) + \frac{2x}{n^s}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{k+j}{n} - x\right) S_{j,k}(x)$$

$$+\frac{1}{n^{2s}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{k+j}{n} - x\right)^2 S_{j,k}(x)$$

$$= x^2 + \frac{x}{n^{2s+1}}.$$

Now, in the same way, one can prove the consequence (iv) immediately.

$$G_{n,s}(t^m;x) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(x + \frac{1}{n^s}\left(\frac{k+j}{n} - x\right)\right)^m S_{j,k}(x)$$

$$= \frac{1}{n^{sm}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left((n^s-1)x + \frac{k+j}{n}\right)^m S_{j,k}(x)$$

$$= \frac{1}{n^{sm}}\sum_{i=0}^{m}\binom{m}{i}((n^s-1)x)^{m-i}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{k+j}{n}\right)^i S_{j,k}(x)$$

$$= \sum_{i=0}^{m}\binom{m}{i}\sum_{\ell=0}^{i}\frac{(n^s-1)^{m-i}}{n^{sm}}\binom{i}{\ell}x^m$$

$$+\sum_{i=0}^{m}\sum_{\ell=0}^{i}\left(\frac{(i-\ell)(i-\ell-1)+\ell(\ell-1)}{2}\right)\frac{(n^s-1)^{m-i}}{n^{sm+1}}\binom{i}{\ell}x^{m-1}$$

$$+ T.L.P.(x). \blacksquare$$

3. Main Results

First, the convergent theorem for the sequence in (3.1) is proved, i.e.,

$G_{n,s}(h(t);x) \to h(x)$ as $n \to \infty$.

Theorem 3.1.

For $x \in [0,\infty)$ there exist a continuous function $h \in C[0,\infty)$ such that

$$\lim_{n\to\infty} G_{n,s}(h(t);x) = h(x) \quad (8)$$

Proof.

From Lemma 2.3, the proof is completed. ■

The next result is the Voronovskaya-type asymptotic formula for the sequence $G_{n,s}(.;x)$.

Theorem 3.2.

Let $h \in C[0,\infty)$ with h'' is existed and continuous, then

$$\lim_{n\to\infty} n^{2s+1} G_{n,s}\big((h(t);x)-h(x)\big)=\frac{x}{2}h''(x). \tag{9}$$

Proof.

By using Taylor's expansion, one has:

$$h(t)=h(x)+h'(x)(t-x)+h''(x)\frac{(t-x)^2}{2}+\varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

$$G_{n,s}(h(t);x)=G_{n,s}(h(x);x)+G_{n,s}(h'(x)(t-x);x)$$

$$+G_{n,s}\left(h''(x)\frac{(t-x)^2}{2};x\right)+G_n(\varepsilon(t,x)(t-x)^2,x).$$

Then,

$$G_{n,s}(h(t);x)=h(x)+\frac{x}{n^{2s+1}}\frac{h''(x)}{2}+G_n(\varepsilon(t,x)(t-x)^2;x)$$

$$\lim_{n\to\infty} n^{2s+1} G_{n,s}\big((h(t);x)-h(x)\big)=\frac{x}{2}h''(x)+\lim_{n\to\infty} n^{2s+1} G_{n,s}(\varepsilon(t,x)(t-x)^2;x)$$

find $\lim\limits_{n\to\infty} n^{2s+1} G_{n,s}(\varepsilon(t,x)(t-x)^2;x)$ required,

$$G_{n,s}(\varepsilon(t,x)(t-x)^2;x)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\varepsilon(t;x)(t-x)^2 S_{j,k}(x)$$

$$n^{2s+1}\left|G_{n,s}(\varepsilon(t,x)(t-x)^2;x)\right| \leq n^{2s+1}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S_{j,k}(x)\,|\varepsilon(t;x)|(t-x)^2$$

$$= n^{2s+1}\underbrace{\sum\sum S_{j,k}(x)|\varepsilon(t;x)|(t-x)^2}_{\left|\frac{k+j}{n}-x\right|<\delta} + n^{2s+1}\underbrace{\sum\sum S_{j,k}(x)|\varepsilon(t;x)|(t-x)^2}_{\left|\frac{k+j}{n}-x\right|\geq\delta}$$

$$:= E_1 + E_2.$$

For each $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$E_1 = n^{2s+1}\underbrace{\sum\sum S_{j,k}(x)|\varepsilon(t;x)|(t-x)^2}_{\left|\frac{k+j}{n}-x\right|<\delta} < n^{2s+1}\varepsilon T_{n,2}(x).$$

Since ε is arbitrary one gets $E_1 = 0$.

$$E_2 = n^{2s+1}\underbrace{\sum\sum S_{j,k}(x)|\varepsilon(t;x)|(t-x)^2}_{\left|\frac{k+j}{n}-x\right|\geq\delta}$$

$$\leq n^{2s+1}\underbrace{\sum\sum S_{j,k}(x)|\varepsilon(t;x)|(t-x)^2}_{\left|\frac{k+j}{n}-x\right|\geq\delta}.$$

By applying Schwarz inequality for summation, one has,

$$E_2 \leq n^{2s+1}C\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S_{j,k}(x)\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} S_{j,k}(x)\,e^{2\alpha t}(t-x)^4\right)^{\frac{1}{2}}.$$

used Taylor's expansion of $e^{\alpha t}$to arrive at,

$$\lim_{n\to\infty} n^{2s+1}G_{n,s}(\varepsilon(t,x)(t-x)^2;x) \to 0 \qquad as\; n \to \infty. \qquad \blacksquare$$

Definition 3.1. [7]

For $h \in C_\alpha[0,\infty)$ the space of uniformly continuous with $\delta > 0$, and $x \in [0,\infty)$, the modulus of continuity for the function $h(x)$ which denoted by $\omega(h;\delta)$ and defined as

$$\omega(\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} |h(x)-h(y)|. \qquad (10)$$

The modulus of continuity is expressed as follows:

$$|h(x)-h(y)| \le \omega(h;\delta)\left(\frac{|x-y|}{\delta}+1\right). \qquad (11)$$

The error estimation of the approximation of the sequence $G_{n,s}(h;x)$ by using the modulus of continuity and the norm of the functions $h, h', h"$ is provided by the following theorem.

Theorem 3.2.

For $\eta > 0$, let $h, h^{(2)} \in C_\alpha[0,\infty)$ exists and continues on $(a-\eta, b+\eta) \subset (0,\infty)$, for some $\alpha > 0$ and $0 \le q \le 2$ then for sufficiently large n,

$$\left\|G_{n,s}(h;x)-h(x)\right\|_{C[a,b]} \le C_1 n^{-1} \sum_{i=0}^{2} \left\|h^{(i)}\right\|_{C[a,b]}$$

$$+C_2\omega_{h^{(q)}}\left(n^{-\frac{1}{2}};(a-\eta,b+\eta)\right)+O(n^{-2}), \qquad (12)$$

where C_1, C_2 are constants independent of h and n.

Proof.

$$h(t)=\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}(t-x)^i+\frac{h^{(q)}(\xi)-h^{(q)}(x)}{q!}(t-x)^q\varkappa(t)+h(t,x)\big(1-\varkappa(t)\big),$$

$\varkappa(t)$ is the characteristic function of the interval$(a-\eta, b+\eta)$, where ξ lies between t, x.

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, one has

$$h(t)=\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}(t-x)^i+\frac{h^{(q)}(\xi)-h^{(q)}(x)}{q!}(t-x)^q.$$

For $t\in[0,\infty)\backslash(a-\eta,b+\eta)$ and $x\in[a,b]$, $g(t,x)$ is described by,

$$g(t,x)=h(t)-\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$G_{n,s}(h,x)-h(x)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\,h(t)-h(x)$$

$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}(t-x)^i-h(x)$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\frac{h^{(q)}(\xi)-h^{(q)}(x)}{q!}(t-x)^q\varkappa(t)+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\,g(t,x)\big(1-\varkappa(t)\big)$$

$$\coloneqq E_1+E_2+E_3.$$

The estimation of the term E_1 is going as:

$$E_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}(t-x)^i-h(x)$$

$$E_1=\sum_{i=0}^{q}\frac{h^{(i)}(x)}{i!}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_{j,k}(x)\sum_{\ell=0}^{i}\binom{i}{\ell}t^{\ell}\,(-x)^{i-\ell}-h(x)$$

and from Lemma 2.3,

$\|E_1\|_{C[a,b]}\le C_1\,n^{-1}\left\|h^{(i)}(x)\right\|$, uniformly on $[a,b]$.

To estimate E_2 we proceed as follows:

$$|E_2| \leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x)\frac{\left|h^{(q)}(\xi)-h^{(q)}(x)\right|}{q!}|t-x|^q\varkappa(t)$$

and from Definition 3.1.

$$|E_2| \leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x)\frac{\omega_{h^{(q)}}\big(\delta;(a-\eta,b+\eta)\big)}{q!}\left(\frac{|t-x|}{\delta}+1\right)|t-x|^q$$

$$|E_2| \leq \frac{\omega_{h^{(q)}}\big(\delta;(a-\eta,b+\eta)\big)}{q!}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x)\,\delta^{-1}|t-x|^{q+1}+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x)\,|t-x|^q\right),$$

$\delta > 0$

By applying Schwarz inequality for summation, one has,

choosing $\delta = n^{-\frac{1}{2}}$, we are led to.

$\|E_2\|_{C[a,b]} \leq C_2\omega_{h^{(q)}}\left(n^{-\frac{1}{2}};(a-\eta,b+\eta)\right).$

since $t \in [0,\infty)/(a-\eta,b+\eta)$ we can choose $\delta > 0$ such that $|t-x| \geq \delta$ for all $x \in [a,b]$.

$$|E_3| \leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_{j,k}(x)\,|h(t,x)|.$$

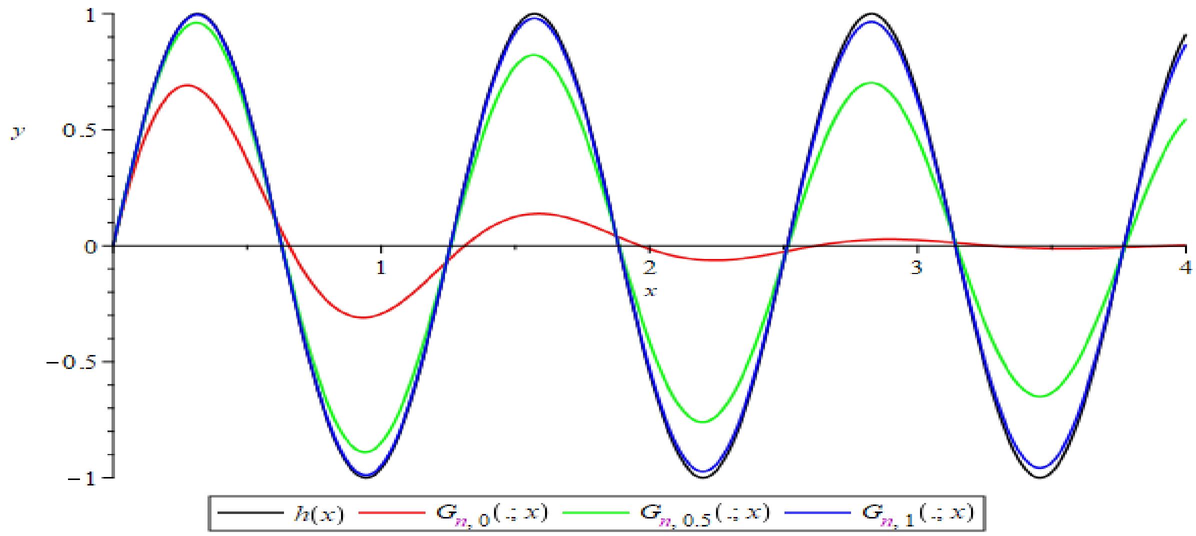
for $|t-x| \geq \delta$ one can find $C > 0$ such that $|h(t,x)| \leq Ce^{\alpha t}$

and by using Taylor's expansion of $e^{\alpha t}$, one has

$\|E_3\|_{C[a,b]} = O(n^{-2}).$ ■

4. Numerical Example

In this example, the test function $h(x) = \sin(5x)$, in the black color is compared to the sequence $G_{n,0}(h;x)$ in red color, $G_{n,0.5}(h;x)$ in green color, and $G_{n,1}(h_1;x)$ in blue color for $n = 100$, actually, when n, s are increased, the approximation $G_{n,s}(h;x)$ yields more accurate numerical results.



5. Conclusions.

The purpose of this article is to generalize linear positive operators, which are introduced and analyzed as a version of the Szasz sequence in double sums. The study also compares the polynomials, $G_{n,s}(.\,;x)$ to the test function. This numerical convergence is shown by the graphs of the, $G_{n,s}(.\,;x)$ with the function $f(x)$ who is best than the classical Szasz sequence.

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الجمع المزدوج لمتتابعة زاز

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المستخلص

قدم هذا البحث ودرس الجمع المزدوج لمتتابعة زاز في متغير واحد. حيث تمت دراسة العلاقة التكرارية من الرتبة m ، ومبرهنة التقارب، ومبرهنة فورونوفسكايا في التقريب، وتم تقديم امثلة عددية لبيان تقارب المتتابعة الجديدة الى الدالة الاختبارية المختارة $sin5x$. أظهرت النتائج العددية افضلية المتتابعات الجديدة بتقريب دالة الاختبار مقارنة بالنتائج العددية للمتتابعة الاعتيادية على الرغم من امتلاكهما نفس رتبة التقريب. تم وصف النتائج العددية برسم بيان دالة الاختبار وتقريباتها الناتجة من متتابعات زاز العادية، الثنائية. حيث نتج من ذلك ان الأفضلية بالنتائج العددية كانت للمتابعة $G_{n,s}(h;x)$.